SUPERSYMMETRIC FIELD THEORIES AND COHOMOLOGY

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Abstract

This is the Ph.D. dissertation of the author. The project has been motivated by the conjecture that the Hopkins-Miller tmf spectrum can be described in terms of 'spaces' of conformal field theories. In this dissertation, spaces of field theories are constructed as classifying spaces of categories whose objects are certain types of field theories. If such a category has a symmetric monoidal structure and its components form a group, by work of Segal, its classifying space is an infinite loop space and defines a cohomology theory. This has been carried out for two classes of field theories: (i) For each $n \in \mathbb{Z}$, there is a category $\mathcal{SEFT}_n$ whose objects are the Stolz-Teichner (1|1)-dimensional super Euclidean field theories of degree n. It is proved that the classifying space $|\mathcal{SEFT}_n|$ represents the degree-n K or KO cohomology, depending on the coefficients of the field theories. (ii) For each $n \in \mathbb{Z}$, there is a category $\mathcal{AFT}_n$ whose objects are a kind of (2|1)-dimensional field theories called 'annular field theories,' defined using supergeometric versions of circles and annuli only. It is proved that the classifying space $|\mathcal{AFT}_n|$ represents the degree-n elliptic cohomology associated with the Tate curve. To the author's knowledge, this is the first time the definitions of low-dimensional supersymmetric field theories are given in full detail.

INTRODUCTION

Background. One of the first connections between (supersymmetric) quantum field theories (QFTs) and topology appeared in Witten's approach to the Atiyah-Singer index theorem. He argued that the circle partition function of a canonical 1-dimensional QFT associated to a compact spin manifold M^{2n} equals $\hat{A}(M)$. [Wit99] (See also [LM89], III, §17.) Later, Ochanine discovered the elliptic genus, which is a bordism invariant of spin manifolds and takes values of integral modular forms. [Och87] Witten identified the elliptic genus as the torus partition function of a 2-dimensional conformal field theory (CFT); furthermore, variations of CFTs give rise to other modular form valued bordism invariants, e.g. the Witten genus. [Wit87] The field-theoretic interpretation of these bordism invariants provides intuitive explanations of some of their properties, e.g. integrality, modularity and rigidity, which can also be proved rigorously. [Seg88, BT89]

Elliptic cohomology theories are cohomology theories where the elliptic genus, the Witten genus, etc., of families take value. Such theories were first constructed by Landweber, Ravenel and Stong, using the Landweber exact functor theorem. [LRS93] There are plenty of elliptic cohomology theories, each associated to an elliptic curve. A lot of work has gone into the search of a 'universal' theory. Ando, Hopkins and Strickland defined and studied elliptic spectra; then Hopkins and Miller constructed the tmf spectrum, which is a suitable inverse limit in the category of elliptic spectra. [AHS01, Hop02] The families

Witten genus, for instance, can now be described as a multiplicative map from a Thom spectrum to tmf. However, these cohomology theories and spectra have been defined using homotopy theory and lack a geometric interpretation.

In [Seg88], Segal proposed that an elliptic cohomology class is represented by something he called an elliptic object. An elliptic object over a point is a CFT, in the sense of [Seg04], and an elliptic object over a general space X can be thought of as 'a CFT parametrized by X.' It has also been conjectured that the tmf spectrum can be recovered in terms of spaces of CFTs. Stolz and Teichner have significantly modified and elaborated upon the precise notions of CFTs and elliptic objects that should be related to tmf; they have also constructed canonical elliptic objects which are expected to represent certain tmf Euler classes. [ST04]

Goal and results. This work focuses on constructing 'spaces' of 1- and 2-dimensional QFTs and relating them to cohomology theories. Following the axiomatic point of view of Atiyah, Segal and Witten, we think of a QFT as a vector space representation of a bordism category. Our approach is to define symmetric monoidal categories whose objects are certain kinds of QFTs, and study their classifying spaces. By [Seg74], these classifying spaces are infinite loop spaces, [1] hence defining cohomology theories.

The main results are as follows.

Theorem 1.2.3. *For each $n \in \mathbb{Z}$, there is a (small) category $\mathcal{SEFT}_n$ whose objects are '1-dimensional super Euclidean field theories of degree n,' such that*

$$|\mathcal{SEFT}_n| \simeq KO_n \text{ or } K_n ,$$

where KO_n (resp. K_n) is the n-th space of the periodic KO-theory (resp. K-theory) spectrum. Whether we have KO_n or K_n depends on the coefficients in the field theories.

Theorem 2.2.2. *For each $n \in \mathbb{Z}$, there is a (small) category $\mathcal{AFT}_n$ whose objects are 'super annular field theories of degree n,' such that*

$$|\mathcal{AFT}_n| \simeq (K_{Tate})_n ,$$

where the right hand side is the n-th space of the elliptic spectrum K_{Tate} associated with the Tate curve.

Here is a section-by-section overview. In §2.1, we define Stolz and Teichner's 1-dimensional super Euclidean field theories as functors between super categories. The use of super categories has been explained and emphasized by Stolz and Teichner. However, to the author's knowledge, its details have never appeared in the literature. §2.2 concerns the definition of $\mathcal{SEFT}_n$. We digress in §2.3 to study another sequence of categories $\mathcal{V}_n$, which are related to $\mathcal{SEFT}_n$, before proving theorem 1.2.3 in §2.4. In §3.1, we construct certain bordism (super) categories using supergeometric versions of S^1 and $S^1 \times [0, t]$, and define 'super annular field theories.' Finally, in §3.2, we define $\mathcal{AFT}_n$ and prove theorem 2.2.2. The proofs of two technical lemmas needed in §2.1 are given in an appendix.

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[1] The components of each of our symmetric monoidal categories form a group.

CHAPTER 1: ONE-DIMENSIONAL FIELD THEORIES AND KO-THEORY

Throughout this chapter, one can replace real coefficients with complex ones and obtain parallel results for K-theory. For simplicity, we focus on KO. *All gradings are* $\mathbb{Z}/2$*-gradings* and all tensor products of graded vector spaces or algebras are graded tensor products. The even and odd parts of a graded vector space V are denoted V^{even} and V^{odd}.

1.1. Stolz and Teichner's One-Dimensional Field Theories

Super manifolds of dimensions $(0|1)$ **and** $(1|1)$**.** We first recall the definition of (smooth) super manifolds. [DM99] A $(p|q)$*-dimensional super manifold* $M = (M_{\text{red}}, \mathcal{O}_M)$ consists of a smooth p-dimensional manifold M_{red}, and a sheaf $\mathcal{O}_M$ of real graded commutative algebras on M_{red}. Any point in M_{red} has a neighborhood U such that $\mathcal{O}_M|_U \cong \mathcal{O}_{M_{\text{red}}}|_U \otimes \Lambda^*\mathbb{R}^q$, where $\mathcal{O}_{M_{\text{red}}}$ is the structure sheaf of M_{red} and $\Lambda^*\mathbb{R}^q$ is the exterior algebra generated by $\mathbb{R}^q$. Elements in $\mathcal{O}(M) := \mathcal{O}_M(M_{\text{red}})$ are refered to as *(global) functions* on M. Examples of super manifolds are $\mathbb{R}^{p|q} = (\mathbb{R}^p, \mathcal{O}_{\mathbb{R}^p} \otimes \Lambda^*\mathbb{R}^q)$. A map $f = (f_{\text{red}}, f^*) : M \to N$ of two super manifolds consists of a map $f_{\text{red}} : M_{\text{red}} \to N_{\text{red}}$ of smooth manifolds and a morphism $f^* : \mathcal{O}_N \to (f_{\text{red}})_*\mathcal{O}_M$ of sheaves of graded commutative algebras extending f^*_{red}.

In this chapter, by a *Euclidean* $(0|1)$*-manifold* (Z, ξ), we mean a $(0|1)$-manifold Z together with a 1-form ξ on Z of the form $\lambda d\lambda$ for some $\lambda \in \mathcal{O}(Z)^{\text{odd}}$ that is nowhere vanishing. [2] On the other hand, a *Euclidean* $(1|1)$*-manifold* (Y, ω) is a $(1|1)$-manifold Y with a 'metric' ω. Following [ST04], a 1-form ω on Y is called a *metric* if every point in Y_{red} has a neighborhood U and $s_U \in \mathcal{O}(Y_U)^{\text{even}}$, $\lambda_U \in \mathcal{O}(Y_U)^{\text{odd}}$, [3] such that ds_U, λ_U are nowhere vanishing and $\omega|_{Y_U} = ds_U + \lambda_U d\lambda_U$. For any Euclidean $(0|1)$- or $(1|1)$-manifolds (M_i, χ_1), $i = 1, 2$, we write $\phi : (M_1, \chi_1) \to (M_2, \chi_2)$ for a map $\phi : M_1 \to M_2$ of super manifolds that satisfies $\chi_1 = \phi^*\chi_2$; the set of these maps is denoted $\text{Hom}(M_1, \chi_1;\, M_2, \chi_2)$.

Consider the Euclidean $(0|1)$-manifold $(\mathbb{R}^{0|1}, \lambda d\lambda)$ and the Euclidean $(1|1)$-manifold $(\mathbb{R}^{1|1}, ds + \lambda d\lambda)$, where λ denotes both the element $1 \in \mathbb{R} = \Lambda^1\mathbb{R} = \mathcal{O}(\mathbb{R}^{0|1})^{\text{odd}}$ and its pullback in $\mathcal{O}(\mathbb{R}^{1|1})^{\text{odd}}$ along the canonical projection $\mathbb{R}^{1|1} \to \mathbb{R}^{0|1}$, while $s \in \mathcal{O}(\mathbb{R}^{1|1})^{\text{even}}$ is the pullback of the identity function on $\mathbb{R}$ along the other canonical projection $\mathbb{R}^{1|1} \to \mathbb{R}$. Define the following maps

$$\begin{aligned} & \epsilon : \mathbb{R}^{0|1} \to \mathbb{R}^{0|1}, \quad \epsilon^*(\lambda) = -\lambda, \\ \text{or} \quad & \epsilon : \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}, \quad \epsilon^*(s, \lambda) = (s, -\lambda), \\ & \gamma_t : \mathbb{R}^{0|1} \to \mathbb{R}^{1|1}, \quad \gamma_t^*(s, \lambda) = (t, \lambda), \quad t \in \mathbb{R} \\ & \tau_t : \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}, \quad \tau_t^*(s, \lambda) = (s + t, \lambda), \quad t \in \mathbb{R}. \end{aligned}$$

Notice that $\gamma_t^*(s) = t$ means $(\gamma_t)_{\text{red}}$ maps $\mathbb{R}^0$ to $\{t\}$; $\tau_t^*(s) = s + t$ means $(\tau_t)_{\text{red}}$ is translation by t on $\mathbb{R}$. We have

$$\text{Hom}(\mathbb{R}^{0|1}, \lambda d\lambda;\, \mathbb{R}^{0|1}, \lambda d\lambda) = \{1, \epsilon\}, \tag{1.1.3}$$
$$\text{Hom}(\mathbb{R}^{0|1}, \lambda d\lambda;\, \mathbb{R}^{1|1}, ds + \lambda d\lambda) = \{\gamma_t, \gamma_t\epsilon\}, \tag{1.1.4}$$
$$\text{Hom}(\mathbb{R}^{1|1}, ds + \lambda d\lambda;\, \mathbb{R}^{1|1}, ds + \lambda d\lambda) = \{\tau_t, \tau_t\epsilon\}. \tag{1.1.5}$$

For example, a map $\tau : \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ is determined by the pair $\tau^*(s, \lambda) = (a(s), b(s)\lambda)$. The condition $\tau^*(ds + \lambda d\lambda) = ds + \lambda d\lambda$ is then equivalent to $a'(s) = 1$ and $b(s)^2 = 1$. This explains (1.1.5).

[2] The notion of differential form extends to smooth super manifolds. See [DM99].
[3] Given a super manifold M and an open subset $U \subset M_{\text{red}}$, we denote by M_U the super submanifold $(U, \mathcal{O}_M|_U)$.

Note. The definition of a Euclidean (1|1)-manifold (Y, ω) can be restated as follows: every point in Y_{red} has a neighhorhood U and a map $\phi : (Y_U, \omega|_{Y_U}) \to (\mathbb{R}^{1|1}, ds + \lambda d\lambda)$. Indeed, suppose $\omega|_{Y_U} = ds_U + \lambda_U d\lambda_U$ for some $s_U \in \mathcal{O}(Y_U)^{\mathrm{even}}$, $\lambda_U \in \mathcal{O}(Y_U)^{\mathrm{odd}}$. The map $\phi = (\phi_{\mathrm{red}}, \phi^*)$ can then be defined by $\phi_{\mathrm{red}} = s_U : U \to \mathbb{R}$ and $\phi^*(\lambda) = \lambda_U$.

Euclidean (0|1)- and (1|1)-manifolds have the following alternative descriptions:

Lemma 1.1.6. *A Euclidean* (0|1)*-manifold* (Z, ξ) *with an orientation on* Z_{red} *is equivalent to a* 0*-manifold* Z_{red} *with a spin structure.*

Proof. Suppose Z_{red} has a spin structure, which consists of an orientation and a real line bundle $S \to Z_{\mathrm{red}}$ with metric. We then obtain a Euclidean (0|1)-manifold (Z, ξ), with $Z = (Z_{\mathrm{red}}, \Gamma(-, \Lambda^* S))$ and $\xi = \lambda d\lambda$ where $\lambda \in \mathcal{O}(Z)^{\mathrm{odd}} = \Gamma(Z_{\mathrm{red}}, S)$ is a section of constant length 1. Notice ξ depends only on the metric of S but not on the choice of λ. Conversely, given a Euclidean (0|1)-manifold (Z, ξ) with $\xi = \lambda d\lambda$ for some $\lambda \in \mathcal{O}(Z)^{\mathrm{odd}}$, we obtain a line bundle $S = \coprod_{x \in Z_{\mathrm{red}}} \mathcal{O}_Z(\{x\})^{\mathrm{odd}}$ over Z_{red} with a metric defined by requiring λ be of constant length 1. This, together with the orientation on Z_{red}, gives a spin structure on Z_{red}. The two constructions above are inverse to each other. □

Lemma 1.1.7. *A Euclidean* (1|1)*-manifold* (Y, ω) *is equivalent to a* 1*-manifold* Y_{red} *with a spin structure.*

Proof. Suppose Y_{red} has a spin structure, which consists of a Riemannian metric, an orientation and a $\mathbb{Z}/2$-bundle $P \to Y_{\mathrm{red}}$. Let $S^+ = P \times_{\mathbb{Z}/2} \mathbb{R}$ be the associated positive spinor bundle with the canonical metric. We first obtain a (1|1)-manifold $Y = (Y_{\mathrm{red}}, \Gamma(-, \Lambda^* S^+))$; a metric ω on Y is then defined by demanding $\omega|_{Y_U} = ds_U + \lambda_U d\lambda_U$ in any open set $U \subset Y_{\mathrm{red}}$ that admits a function s_U measuring the arc length in the positive direction and a section $\lambda_U \in \Gamma(U, S^+) = \mathcal{O}(Y_U)^{\mathrm{odd}}$ of constant length 1. Such ω exists and is unique. Conversely, given a Euclidean (1|1)-manifold (Y, ω), pick an open cover $\{U_\alpha\}$ of Y_{red} and maps $f_\alpha : (Y_{U_\alpha}, \omega|_{Y_\alpha}) \to (\mathbb{R}^{1|1}, ds + \lambda d\lambda)$. If $U_\alpha \cap U_\beta \neq \emptyset$, the change of coordinates $\phi_{\beta\alpha} = f_\beta f_\alpha^{-1}$ preserves $ds + \lambda d\lambda$. By (1.1.5), $\phi^*_{\beta\alpha}(ds) = ds$ and $\phi^*_{\beta\alpha}(\lambda) = \sigma_{\beta\alpha}\lambda$ for some $\sigma_{\beta\alpha} = \pm 1$. The nowhere vanishing 1-forms $\{f^*_\alpha(ds)\}$ consistently induce a Riemannian metric and an orientation on Y_{red}; the cocycle $\{\sigma_{\beta\alpha}\}$ defines a $\mathbb{Z}/2$-bundle over Y_{red}. These constitute a spin structure on Y_{red}. The two constructions above are inverse to each other. □

Remark. Despite these equivalences, in most of the discussions involving Euclidean (0|1)- and (1|1)-manifolds, we view them as super manifolds.

Let us interpret functions and 1-forms on a super manifold M in terms of its functor of points: super manifold $B \mapsto M(B) := \{\text{maps from } B \text{ to } M\}$. Associated to any $a \in \mathcal{O}(M)$ is a natural transformation

$$a_B : M(B) \to \mathcal{O}(B), \quad f \mapsto f^* a. \tag{1.1.8}$$

Also, each 1-form on M of the form adb, where $a, b \in \mathcal{O}(M)$, induces a natural transformation as follows

$$(adb)_B : TM(B) \to \mathcal{O}(B), \quad (f; \delta f) \mapsto (f^* a) \left(\frac{d}{dt} f_t^* b \right) \bigg|_{t=0}, \tag{1.1.9}$$

where $\{f_t\} \subset M(B)$ is a smooth one-parameter family whose tangent at $t = 0$ equals $(f; \delta f)$. [4] For instance, the elements $\lambda \in \mathcal{O}(\mathbb{R}^{0|1})^{\mathrm{odd}} \subset \mathcal{O}(\mathbb{R}^{1|1})^{\mathrm{odd}}$, $s \in \mathcal{O}(\mathbb{R}^{1|1})^{\mathrm{even}}$ defined above induce the following bijections

$$\lambda_B : \mathbb{R}^{0|1}(B) \xrightarrow{\sim} \mathcal{O}(B)^{\mathrm{odd}}, \qquad (s_B, \lambda_B) : \mathbb{R}^{1|1}(B) \xrightarrow{\sim} \mathcal{O}(B)^{\mathrm{even}} \times \mathcal{O}(B)^{\mathrm{odd}}. \tag{1.1.10}$$

[4] In our situations below, $M(B)$ always has the structure of a smooth, finite dimensional vector bundle over a smooth manifold, hence in particular has a smooth structure.

From now on, we will identify $\mathbb{R}^{0|1}(B)$ with $\mathcal{O}(B)^{\text{odd}}$ and $\mathbb{R}^{1|1}(B)$ with $\mathcal{O}(B)^{\text{even}} \times \mathcal{O}(B)^{\text{odd}}$ using these bijections. The 1-form $\lambda d\lambda$ on $\mathbb{R}^{0|1}$ defines the maps

$$(\lambda d\lambda)_B : T\mathbb{R}^{0|1}(B) \to \mathcal{O}(B), \quad (\eta; \delta\eta) \mapsto \eta\delta\eta.$$

We have used (1.1.10), so that $\eta \in \mathcal{O}(B)^{\text{odd}}$, $\delta\eta \in T_\eta\mathcal{O}(B)^{\text{odd}} \cong \mathcal{O}(B)^{\text{odd}}$ and $\lambda_B(\eta) = \eta$. Similarly, the 1-form $ds + \lambda d\lambda$ on $\mathbb{R}^{1|1}$ defines the maps

$$(ds + \lambda d\lambda)_B : T\mathbb{R}^{1|1}(B) \to \mathcal{O}(B), \quad (w, \eta; \delta w, \delta\eta) \mapsto \delta w + \eta\delta\eta,$$

where $w, \delta w \in \mathcal{O}(B)^{\text{even}}$.

Given Euclidean (0|1)- or (1|1)-manifolds (M_i, χ_i), $i = 1, 2$, we use the notation $\phi : \big(M_1(B), (\chi_1)_B\big) \to \big(M_2(B), (\chi_2)_B\big)$ for a smooth map $\phi : M_1(B) \to M_2(B)$ that satisfies $(\chi_1)_B = (\chi_2)_B \circ T\phi$; we write $\mathrm{Hom}\big(M_1(B), (\chi_1)_B; M_2(B), (\chi_2)_B\big)$ for the set of such maps. For example, keeping (1.1.10) in mind, we define

$$\begin{aligned} &\epsilon : \mathbb{R}^{0|1}(B) \to \mathbb{R}^{0|1}(B), \quad \epsilon(\eta) = -\eta, \\ \text{or} \qquad &\epsilon : \mathbb{R}^{1|1}(B) \to \mathbb{R}^{1|1}(B), \quad \epsilon(w, \eta) = (w, -\eta), \\ &\gamma_{z,\theta} : \mathbb{R}^{0|1}(B) \to \mathbb{R}^{1|1}(B), \quad \gamma_{z,\theta}(\eta) = (z - \theta\eta, \theta + \eta), \\ &\tau_{z,\theta} : \mathbb{R}^{1|1}(B) \to \mathbb{R}^{1|1}(B), \quad \tau_{z,\theta}(w, \eta) = (w + z - \theta\eta, \theta + \eta), \end{aligned}$$

where $(z, \theta) \in \mathcal{O}(B)^{\text{even}} \times \mathcal{O}(B)^{\text{odd}}$. Then, we have

$$\mathrm{Hom}\big(\mathbb{R}^{0|1}(B), (\lambda d\lambda)_B; \mathbb{R}^{0|1}(B), (\lambda d\lambda)_B\big) = \{1, \epsilon\}, \tag{1.1.11}$$

$$\mathrm{Hom}\big(\mathbb{R}^{0|1}(B), (\lambda d\lambda)_B; \mathbb{R}^{1|1}(B), (ds + \lambda d\lambda)_B\big) = \{\gamma_{z,\theta}, \gamma_{z,\theta}\epsilon\}, \tag{1.1.12}$$

$$\mathrm{Hom}\big(\mathbb{R}^{1|1}(B), (ds + \lambda d\lambda)_B; \mathbb{R}^{1|1}(B), (ds + \lambda d\lambda)_B\big) = \{\tau_{z,\theta}, \tau_{z,\theta}\epsilon\}. \tag{1.1.13}$$

To obtain, say (1.1.13), we suppose a map $\tau : \mathbb{R}^{1|1}(B) \to \mathbb{R}^{1|1}(B)$ has the form $\tau(w, \eta) = (a(w) + b(w)\eta, c(w) + d(w)\eta)$, apply both sides of $(ds + \lambda d\lambda)_B = (ds + \lambda d\lambda)_B \circ T\tau$ to $(w, \eta; \delta w, \delta\eta) \in T\mathbb{R}^{0|1}(B)$, and solve the resulting equation for $a(w)$, $b(w)$, $c(w)$ and $d(w)$. For any B, there is a natural map

$$\begin{array}{ccc} \mathrm{Hom}(M_1, \chi_1; M_2, \chi_2) & \to & \mathrm{Hom}\big(M_1(B), (\chi_1)_B; M_2(B), (\chi_2)_B\big) \\ \phi & \mapsto & (\phi_B : f \mapsto \phi \circ f). \end{array} \tag{1.1.14}$$

In particular, (1.1.3) $\to$ (1.1.11) is a bijection, while (1.1.4) $\to$ (1.1.12) is given by $\gamma_t \mapsto \gamma_{t,0}$, $\gamma_t\epsilon \mapsto \gamma_{t,0}\epsilon$, and (1.1.5) $\to$ (1.1.13) by $\tau_t \mapsto \tau_{t,0}$, $\tau_t\epsilon \mapsto \tau_{t,0}\epsilon$. Although (1.1.14) is not always surjective, we have the following result, as well as lemma 1.1.27.

Lemma 1.1.15. *Suppose (M_i, χ_i), $i = 1, 2$, are Euclidean (0|1)- or (1|1)-manifolds, and B is a super manifold with $B_{red} = \{b\}$. Given a map*

$$\phi : \big(M_1(B), (\chi_1)_B\big) \to \big(M_2(B), (\chi_2)_B\big),$$

there exists a map $\phi_{red} : (M_1)_{red} \to (M_2)_{red}$ of ordinary manifolds so that the diagram

$$\begin{array}{ccc} \big(M_1(B), (\chi_1)_B\big) & \xrightarrow{\phi} & \big(M_2(B), (\chi_2)_B\big) \\ \downarrow & & \downarrow \\ (M_1)_{red} & \xrightarrow{\phi_{red}} & (M_2)_{red}\,, \end{array}$$

with the vertical arrows given by $f \mapsto f_{red}(b)$, commutes. In other words, $\phi(f)_{red} = \phi_{red} \circ f_{red}$, $\forall f \in M_1(B)$. Furthermore, $(\phi'\phi)_{red} = \phi'_{red} \circ \phi_{red}$ whenever ϕ and ϕ' are composable.

Proof. See appendix A. □

Super Euclidean field theories of degree zero. Let $\mathcal{S}$ be the category whose objects are super manifolds B with $B_{\text{red}} =$ a point, and whose morphisms are maps of super manifolds. In this chapter, a *super category* $\mathcal{C}$ is a functor $\mathcal{C} : \mathcal{S}^{\text{op}} \to \mathbf{CAT}$, where $\mathbf{CAT}$ is the category of categories and functors. [5] A *functor of super categories* $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a natural transformation between $\mathcal{C}_1$ and $\mathcal{C}_2$. We will now give two examples of super categories.

The first super category we define is $\mathcal{S}$Hilb. Consider the category $\mathcal{S}\text{Hilb}(B)$ associated with an object B of $\mathcal{S}$. The objects of $\mathcal{S}\text{Hilb}(B)$, which are independent of B, are real graded Hilbert spaces. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two of them. The set of morphisms from $\mathcal{H}_1$ to $\mathcal{H}_2$ in $\mathcal{S}\text{Hilb}(B)$ is the graded tensor product $\mathcal{O}(B) \otimes B(\mathcal{H}_1, \mathcal{H}_2)$, where the grading on $B(\mathcal{H}_1, \mathcal{H}_2)$ is induced by those of $\mathcal{H}_1$ and $\mathcal{H}_2$. Composition is defined by composition of operators and the algebra structure on $\mathcal{O}(B)$. Given a morphism $f : B \to B'$ in $\mathcal{S}$, the functor $\mathcal{S}\text{Hilb}(f) : \mathcal{S}\text{Hilb}(B') \to \mathcal{S}\text{Hilb}(B)$ is defined to be identity on objects and $f^* \otimes 1 : \mathcal{O}(B') \otimes B(\mathcal{H}_1, \mathcal{H}_2) \to \mathcal{O}(B) \otimes B(\mathcal{H}_1, \mathcal{H}_2)$ on morphisms.

The second super category $\mathcal{SEB}^1$ is a 'super' version of a $(0+1)$-dimensional bordism category. Fix an object B of $\mathcal{S}$ and let $\Lambda = \mathcal{O}(B)$. We now describe the category $\mathcal{SEB}^1(B)$. The objects of $\mathcal{SEB}^1(B)$ are independent of B. For each pair of nonnegative integers m and n, there is an object $Z_{m,n}$, which is the super manifold $\underbrace{\mathbb{R}^{0|1} \sqcup \cdots \sqcup \mathbb{R}^{0|1}}_{m+n}$ with its first m underlying points positively oriented and the rest negative oriented. Let $\lambda \in \mathcal{O}(Z_{m,n})^{\text{odd}}$ be the element whose restriction to each $\mathbb{R}^{0|1}$ equals what we also denote by λ in e.g. (1.1.3). There are two types of morphisms in $\mathcal{SEB}^1(B)$, 'B-isomorphisms' and equivalence classes of 'B-bordisms.'

A *B-isomorphism*, always a morphism from some $Z_{m,n}$ to itself, is a map

$$\big(Z_{m,n}(B), (\lambda d\lambda)_B\big) \xrightarrow{\mu} \big(Z_{m,n}(B), (\lambda d\lambda)_B\big),$$

such that μ_{red} (see lemma 1.1.15) is bijective and orientation-preserving. For example, according to (1.1.11), the B-isomorphisms on $Z_{1,0}$ are 1 and ϵ.

A *B-bordism* from $Z_{m,n}$ to $Z_{p,q}$ is a 4-tuple $(Y, \omega, \alpha, \beta)$, consisting of a Euclidean $(1|1)$-manifold (Y, ω) with Y_{red} compact, and two maps

$$\begin{tikzcd}
 & \big(Y(B), \omega_B\big) & \\
\big(Z_{m,n}(B), (\lambda d\lambda)_B\big) \arrow[ur, "\alpha"] & & \big(Z_{p,q}(B), (\lambda d\lambda)_B\big). \arrow[ul, "\beta"']
\end{tikzcd}$$

By lemma 1.1.7, ω induces an orientation on Y_{red}, which in turn induces an orientation on ∂Y_{red}. We require $\alpha_{\text{red}} \cup \beta_{\text{red}}$ map $(Z_1)_{\text{red}} \cup (Z_2)_{\text{red}}$ bijectively onto ∂Y_{red} with α_{red} orientation-reversing and β_{red} orientation-preserving. (See lemma 1.1.15.) Two B-bordisms $(Y, \omega, \alpha, \beta)$ and $(Y', \omega', \alpha', \beta')$ are *equivalent*

[5] A super category may also be defined as a contravariant functor from the category of *all* super manifolds to $\mathbf{CAT}$. The use of super categories in defining field theories is emphasized by Stolz and Teichner. [ST04]

if there is an invertible map $\tau : \big(Y(B), \omega_B\big) \to \big(Y'(B), \omega'_B\big)$ making the following diagram

$$\begin{tikzcd}
& Y(B) \arrow[dd, "\tau"] & \\
Z_{m,n}(B) \arrow[ur, "\alpha"] \arrow[dr, "\alpha'"'] & & Z_{p,q}(B) \arrow[ul, "\beta"'] \arrow[dl, "\beta'"] \\
& Y'(B) &
\end{tikzcd} \tag{1.1.16}$$

commute. We denote the equivalence class of $(Y, \omega, \alpha, \beta)$ by $[Y, \omega, \alpha, \beta]$ and *identify equivalent B-bordisms as the same morphism* in $\mathcal{SEB}^1(B)$.

As an example, let us find all B-bordisms between $Z_{1,0}$ and itself of the form $(\mathbb{R}^{1|1}_{[0,t]}, ds + \lambda d\lambda, \alpha, \beta)$, where $t > 0$, and classify them up to equivalence. Notice that

$$\mathbb{R}^{1|1}_{[0,t]}(B) = \Lambda^{\text{even}}_{[0,t]} \times \Lambda^{\text{odd}},$$

where, for any $S \subset \mathbb{R}$, we denote by $\Lambda^{\text{even}}_S \subset \Lambda^{\text{even}}$ the subset of elements whose Λ^0-parts lie in S. By (1.1.12), we have $\alpha = \gamma_{z_1,\theta_1}$ or $\gamma_{z_1,\theta_1}\epsilon$ and $\beta = \gamma_{z_2,\theta_2}$ or $\gamma_{z_2,\theta_2}\epsilon$, where $z_1 \in \Lambda^{\text{even}}_{\{0\}}$, $z_2 \in \Lambda^{\text{even}}_{\{t\}}$, $\theta_1, \theta_2 \in \Lambda^{\text{odd}}$. Let τ be as in (1.1.16) with $(Y, \omega) = (Y', \omega') = (\mathbb{R}^{1|1}_{[0,t]}, ds + \lambda d\lambda)$. By (1.1.13), $\tau = \tau_{z,\theta}$ or $\tau_{z,\theta}\epsilon$ with $(z, \theta) \in \Lambda^{\text{even}}_{\{0\}} \times \Lambda^{\text{odd}}$. Observe that, given $\alpha = \gamma_{z_1,\theta_1}$ (resp. $\gamma_{z_1,\theta_1}\epsilon$), there is a unique τ, namely $\tau_{-z_1,-\theta_1}$ (resp. $\tau_{-z_1,-\theta_1}\epsilon$), such that $\alpha' = \tau\alpha = \gamma_{0,0}$. Therefore, up to equivalence, we may assume $\alpha = \gamma_{0,0}$, while β can be any γ_{z_2,θ_2} or $\gamma_{z_2,\theta_2}\epsilon$. We define

$$\begin{aligned} I_{z,\theta} &= [\,\mathbb{R}^{1|1}_{[0,z^0]}, ds + \lambda d\lambda, \gamma_{0,0}, \gamma_{z,\theta}\,] \\ \epsilon I_{z,\theta} &= [\,\mathbb{R}^{1|1}_{[0,z^0]}, ds + \lambda d\lambda, \gamma_{0,0}, \gamma_{z,\theta}\epsilon\,] \end{aligned} \tag{1.1.17}$$

where now $z \in \Lambda^{\text{even}}_{(0,\infty)}$, $\theta \in \Lambda^{\text{odd}}$ and z^0 is the Λ^0-part of z. According to the definition of composition to be given below, $\epsilon I_{z,\theta}$ is indeed the composition of $I_{z,\theta}$ and ϵ.

To finish the definition of $\mathcal{SEB}^1(B)$, it remains to describe how morphisms compose (μ, μ' are B-isomorphisms):

$$\begin{array}{lcl} \mu' \circ \mu & = & \mu'\mu \\ \mu' \circ [Y, \omega, \alpha, \beta] & = & [Y, \omega, \alpha, \beta\mu'^{-1}] \\ [Y', \omega', \alpha', \beta'] \circ \mu & = & [Y', \omega', \alpha'\mu, \beta'] \\ [Y', \omega', \alpha', \beta'] \circ [Y, \omega, \alpha, \beta] & = & [Y'', \omega'', \iota\alpha, \iota'\beta']. \end{array}$$

The last case is described in a commutative diagram (the 1-forms suppressed)

$$\begin{tikzcd}
& & Y''(B) & & \\
& Y(B) \arrow[ur, "\iota"] & & Y'(B) \arrow[ul, "\iota'"'] & \\
Z_{m,n}(B) \arrow[ur, "\alpha"] & & Z_{p,q}(B) \arrow[ul, "\beta"'] \arrow[ur, "\alpha'"] & & Z_{r,s}(B), \arrow[ul, "\beta'"']
\end{tikzcd} \tag{1.1.18}$$

where ι and ι' are injections making the square in the above diagram a pushout. It is straightforward to check that $(Y'', \omega'', \iota\alpha, \iota'\beta')$ is well-defined up to equivalence.

Let us compute the relations among various endomorphisms of $Z_{1,0}$. Clearly, $\epsilon^2 = 1$. Let $Y = \mathbb{R}^{1|1}_{[0,z^0]}$ and $\omega = ds + \lambda d\lambda$. Consider

$$\begin{aligned} \epsilon I_{z,\theta}\epsilon &= \epsilon \circ [\,Y, \omega, \gamma_{0,0}, \gamma_{z,\theta}\,] \circ \epsilon = [\,Y, \omega, \gamma_{0,0}\epsilon, \gamma_{z,\theta}\epsilon\,] \\ &= [\,Y, \omega, \epsilon\gamma_{0,0}\epsilon, \epsilon\gamma_{z,\theta}\epsilon\,] = [\,Y, \omega, \gamma_{0,0}, \gamma_{z,-\theta}\,] \\ &= I_{z,-\theta}\ , \end{aligned} \tag{1.1.19}$$

where the third equality follows from the definition of equivalent B-bordisms and the fourth equality is an easy computation. On the other hand, the commutative diagram

$$\begin{tikzcd}
 & & & \Lambda^{\mathrm{even}}_{[0,z^0+z'^0]} \times \Lambda^{\mathrm{odd}} & & & \\
 & \Lambda^{\mathrm{even}}_{[0,z^0]} \times \Lambda^{\mathrm{odd}} \arrow[urr, "\tau_{0,0}"] & & & & \Lambda^{\mathrm{even}}_{[0,z'^0]} \times \Lambda^{\mathrm{odd}} \arrow[ull, "\tau_{z,\theta}"'] & \\
\Lambda^{\mathrm{odd}} \arrow[ur, "\gamma_{0,0}"] & & & \Lambda^{\mathrm{odd}} \arrow[ull, "\gamma_{z,\theta}"'] \arrow[urr, "\gamma_{0,0}"] & & & \Lambda^{\mathrm{odd}}\ , \arrow[ul, "\gamma_{z',\theta'}"']
\end{tikzcd}$$

together with the facts $\tau_{0,0}\gamma_{0,0} = \gamma_{0,0}$ and $\tau_{z,\theta}\gamma_{z',\theta'} = \gamma_{z+z'-\theta\theta',\theta+\theta'}$, shows

$$I_{z',\theta'} I_{z,\theta} = I_{z+z'-\theta\theta',\theta+\theta'}. \tag{1.1.20}$$

This defines the structure of a semigroup on $\{I_{z,\theta}\} \cong \Lambda^{\mathrm{even}}_{(0,\infty)} \times \Lambda^{\mathrm{odd}}$. [6]

Finally, given a morphism $f : B' \to B$ in $\mathcal{S}$, we describe the functor $\mathcal{SEB}^1(f) : \mathcal{SEB}^1(B) \to \mathcal{SEB}^1(B')$. It is identity on objects and B-isomorphisms. Notice that any B-isomorphism is an orientation-preserving permutation of points together with a choice of 1 or ϵ for each point. The set of B-isomorphisms is thus 'the same' for all B and it makes sense to say $\mathcal{SEB}^1(f)$ is identity on them. As to B-bordisms, we first define for any $(z, \theta) \in \Lambda^{\mathrm{even}}_{(0,\infty)} \times \Lambda^{\mathrm{odd}}$ the following:

$$\begin{aligned} \bar{I}_{z,\theta} &= [\,\mathbb{R}^{1|1}_{[0,z^0]}, ds + \lambda d\lambda, \gamma_{z,\theta}, \gamma_{0,0}\,]\ , \\ LI_{z,\theta} &= [\,\mathbb{R}^{1|1}_{[0,z^0]}, ds + \lambda d\lambda, \gamma_{0,0} \sqcup \gamma_{z,\theta}, -\,]\ , \\ RI_{z,\theta} &= [\,\mathbb{R}^{1|1}_{[0,z^0]}, ds + \lambda d\lambda, -, \gamma_{z,\theta} \sqcup \gamma_{0,0}\,]\ , \end{aligned} \tag{1.1.21}$$

which are respectively B-bordisms from $Z_{0,1}$ to $Z_{0,1}$, from $Z_{1,1}$ to $Z_{0,0}$, and from $Z_{0,0}$ to $Z_{1,1}$. A general B-bordism is a disjoint union of ones of the form $I_{z,\theta}$, $\bar{I}_{z,\theta}$, $LI_{z,\theta}$, $RI_{z,\theta}$ and $[Y, \omega, -, -]$ where $Y_{\mathrm{red}} = S^1$. The map $\mathcal{SEB}^1(f)$ on B-bordisms is defined by $I_{z,\theta} \mapsto I_{f^*z, f^*\theta}$ (and similarly for $\bar{I}_{z,\theta}$, $LI_{z,\theta}$ and $RI_{z,\theta}$), $[Y, \omega, -, -] \mapsto [Y, \omega, -, -]$, and the requirement that it preserves disjoint unions.

Definition 1.1.22. A *one-dimensional super Euclidean field theory*, or 1-*SEFT*, of degree 0 is a functor $E : \mathcal{SEB}^1 \to \mathcal{S}\mathrm{Hilb}$ of super categories. Let $E_B : \mathcal{SEB}^1(B) \to \mathcal{S}\mathrm{Hilb}(B)$ be the (ordinary) functor

[6] Furthermore, as B varies, we obtain a contravariant functor from $\mathcal{S}$ to the category of semigroups, i.e. a *super semigroup*. [DM99]

associated with each object B of $\mathcal{S}$. Since on objects E_B does not depend on B, we simply write $E(Z_{m,n})$. Let $\mathcal{H} = E(Z_{1,0})$. A priori $E_B(\epsilon) \in \mathcal{O}(B) \otimes B(\mathcal{H})$, but since $(f^* \otimes 1)E_{B'}(\epsilon) = E_B(\epsilon)$ for every morphism $f : B \to B'$ in $\mathcal{S}$, each $E_B(\epsilon)$ must in fact be an element of $B(\mathcal{H})$ independent of B. Abusing notations, we denote that element also by ϵ. Now, fix B and let $\Lambda = \mathcal{O}(B)$. The functor E_B is required to satisfy the following assumptions:

(i) E_B sends $Z_{0,0}$ to $\mathbb{R}$ and disjoint unions of objects (resp. morphisms) to tensor products of graded Hilbert spaces (resp. operators). More precisely, tensor products of operators are maps of the form given by multiplication in Λ:

$$\big(\Lambda \otimes B(\mathcal{H}_1, \mathcal{H}_2)\big) \otimes \big(\Lambda \otimes B(\mathcal{H}'_1, \mathcal{H}'_2)\big) \to \Lambda \otimes B(\mathcal{H}_1 \otimes \mathcal{H}'_1, \mathcal{H}_2 \otimes \mathcal{H}'_2).$$

(ii) E_B sends objects (resp. B-bordisms) with opposite orientations to the same graded Hilbert space (resp. operator), e.g. $E(Z_{n,m}) = E(Z_{m,n})$, $E_B(\bar{I}_{z,\theta}) = E_B(I_{z,\theta})$. [7]

(iii) If $(Y, \omega, \alpha, \beta)$ is a B-bordism from $Z_{m,n}$ to $Z_{p,q}$, then $(Y, \omega, -, \alpha \sqcup \beta)$ is a B-bordism from $Z_{0,0}$ to $Z_{n+p,m+q}$. The map

$$\mathrm{Ad}_{E(Z_{m,n})} : \Lambda \otimes E(Z_{n,m}) \otimes E(Z_{p,q}) \to \Lambda \otimes B\big(E(Z_{m,n}), E(Z_{p,q})\big)$$

given by taking adjoints sends $E_B([Y, \omega, -, \alpha \sqcup \beta])$ to $E_B([Y, \omega, \alpha, \beta])$. We have used the fact $E(Z_{n,m})$ is canonically isomorphic to the dual of $E(Z_{m,n})$. [8]

(iv) The operator $\epsilon = E_B(\epsilon)$ on $\mathcal{H}$ is the grading homomorphism, i.e $\epsilon = 1$ on $\mathcal{H}^{\mathrm{even}}$ and $\epsilon = -1$ on $\mathcal{H}^{\mathrm{odd}}$.

(v) $E_B(I_{z,\theta})$ is analytic in $(z, \theta) \in \Lambda^{\mathrm{even}}_{(0,\infty)} \times \Lambda^{\mathrm{odd}}$.

Let us examine the implications of assumptions (i)-(v). For objects, we have $E(Z_{m,n}) = \mathcal{H}^{\otimes(m+n)}$ by (i) and (ii). Also, (i), (ii) and (iv) determine the action of E_B on B-isomorphisms. Since every B-bordism is a disjoint union of ones of the forms $I_{z,\theta}$, $\bar{I}_{z,\theta}$, $LI_{z,\theta}$, $RI_{z,\theta}$ and $[Y, \omega, -, -]$ with $Y_{\mathrm{red}} = S^1$, and $[Y, \omega, -, -]$ is a composition of the others, it suffices by (i) and functoriality to understand $E_B(I_{z,\theta})$, $E_B(\bar{I}_{z,\theta})$, $E_B(LI_{z,\theta})$ and $E_B(RI_{z,\theta})$. According to (iii), the three adjoint maps

$$\begin{tikzcd}
& \Lambda \otimes \mathcal{H}^{\otimes 2} \arrow[dl, "\mathrm{Ad}^{(1)}_{\mathcal{H}}"'] \arrow[d, "\mathrm{Ad}^{(2)}_{\mathcal{H}}"'] \arrow[dr, "\mathrm{Ad}_{\mathcal{H}^{\otimes 2}}"] & \\
\Lambda \otimes B(\mathcal{H}) \arrow[r, leftrightarrow, "(\cdot)^T"] & \Lambda \otimes B(\mathcal{H}) & \Lambda \otimes (\mathcal{H}^{\otimes 2})^\vee
\end{tikzcd}$$

send $E_B(RI_{z,\theta})$ to $E_B(I_{z,\theta})$, $E_B(\bar{I}_{z,\theta})$ and $E_B(LI_{z,\theta})$, where $\mathrm{Ad}^{(1)}_{\mathcal{H}}$ and $\mathrm{Ad}^{(2)}_{\mathcal{H}}$ correspond to the two different factors of $\mathcal{H}^{\otimes 2}$. Any one of the four operators, say $E_B(I_{z,\theta})$, determines the others because the adjoint maps are injective. Being in the image of $\mathrm{Ad}^{(1)}_{\mathcal{H}}$ precisely means $E_B(I_{z,\theta})$ is Hilbert-Schmidt. Also, the images under $\mathrm{Ad}^{(1)}_{\mathcal{H}}$ and $\mathrm{Ad}^{(2)}_{\mathcal{H}}$ of the same element are transpose of one another, hence $E_B(\bar{I}_{z,\theta}) = E_B(I_{z,\theta})^T$. It then follows by (ii) that $E_B(I_{z,\theta})$ is self-adjoint.

The relations among the morphisms in $\mathcal{SEB}^1(B)$ impose further restrictions on the operators $E_B(I_{z,\theta})$. Let $E_B(I_{z,\theta}) = X(z) + \theta Y(z)$, where $X(z), Y(z) \in \Lambda \otimes B(\mathcal{H})$ are self-adjoint and Hilbert-Schmidt (hence

[7] For complex coefficients, opposite orientation corresponds to complex conjugation.
[8] This is the case for both real and complex coefficients. See footnote 7.

compact). The relation in (1.1.19) and (iv) imply $X(z)$ are even and $Y(z)$ are odd. The composition law (1.1.20) translates into the equations

$$X(z)X(z') - \theta\theta' Y(z)Y(z') = X(z + z' + \theta\theta'), \tag{1.1.23}$$

$$\theta' X(z)Y(z') + \theta Y(z)X(z') = (\theta + \theta')Y(z + z' + \theta\theta'). \tag{1.1.24}$$

By (v), $X(z)$ and $Y(z)$ are determined by their values for, say positive real z. Let t, t' denote positive real numbers. Arguing in the same way we did for ϵ, we see that $X(t)$ and $Y(t)$ are elements of $B(\mathcal{H})$ independent of B. We claim that all $X(t)$ and $Y(t)$ commute. This is true among $X(t)$ by (1.1.23) with $\theta = 0$. Using (1.1.23) again and the commutativity among $X(t)$, we see that $Y(t)Y(t') - Y(t')Y(t) \in B(\mathcal{H})$ is annihilated by any $\theta\theta'$, and so must itself vanish. Finally, using (1.1.24) with $(z, \theta, z', \theta') = (t, 0, t', \theta')$ and $(t', \theta, t, 0)$, we obtain $X(t)Y(t') = Y(t')X(t)$. This proves our claim. By the spectral theorem, the commuting family of self-adjoint compact operators $\{X(t), Y(t)\}$ has a simultaneous eigenspace decomposition. Since $X(t)X(t') = X(t + t')$ for all $t, t' > 0$, there is an even, self-adjoint operator H with compact resolvent defined on a graded subspace $\mathcal{H}' \subset \mathcal{H}$, such that $X(t) = e^{-tH}$ on $\mathcal{H}'$ and $X(t) = 0$ on $\mathcal{H}'^{\perp}$. Let $Q = Y(1)e^{H}$, which is an odd, self-adjoint operator on $\mathcal{H}'$ commuting with H. For $t > 1$, (1.1.24) implies $Y(t) = Y(1)X(t-1) = Qe^{-tH}$ on $\mathcal{H}'$, and $Y(t) = 0$ on $\mathcal{H}'^{\perp}$; this in fact holds for all $t > 0$ by analyticity. Equation (1.1.23), restricted to $\mathcal{H}'$, now becomes $e^{-(z+z')H}(1 - \theta\theta' Q^2) = e^{-(z+z'+\theta\theta')H}$, which implies $Q^2 = H$. Therefore, we have

$$E_B(I_{z,\theta}) = X(z) + \theta Y(z) = \begin{cases} e^{-zQ^2} + \theta Q e^{-zQ^2} & \text{on } \mathcal{H}' \\ 0 & \text{on } \mathcal{H}'^{\perp} \end{cases}. \tag{1.1.25}$$

[9] Conversely, (1.1.25) always satisfies (1.1.23) and (1.1.24).

To summarize, a 1-SEFT E of degree 0 is determined by a real graded Hilbert space $\mathcal{H} = E(Z_{1,0})$ and an odd, self-adjoint operator Q with compact resolvent defined on a graded subspace of $\mathcal{H}$. This operator is referred to as the *generator* of E.

Super Euclidean field theories of general degrees. Given a Euclidean (0|1)-manifold (Z, ξ) with Z_{red} oriented, let $C(Z, \xi)$ be the Clifford algebra on $\mathcal{O}(Z)^{\text{odd}}$ with respect to the quadratic form $\lambda' \mapsto \sum_{x \in Z_{\text{red}}} \operatorname{sgn}(x)[\lambda'(x)/\lambda(x)]^2$, where $\lambda \in \mathcal{O}(Z)^{\text{odd}}$ such that $\xi = \lambda d\lambda$, and $\operatorname{sgn}(x) = \pm 1$ is the orientation of x. Notice the quadratic form depends only on ξ but not on the choice of λ. A map $\mu : (Z, \xi) \to (Z', \xi')$ between two such (0|1)-manifolds, with μ_{red} orientation-preserving, induces an algebra homomorphism $\mu^* : C(Z', \xi') \to C(Z, \xi)$. By lemma 1.1.6, (Z, ξ) induces a spin structure on Z_{red}, which includes a real line bundle $S \to Z_{\text{red}}$ with metric $||\cdot||$; $C(Z, \xi)$ can then be described as in [ST04], i.e. the Clifford algebra on $\Gamma(Z_{\text{red}}, S)$ with respect to the quadratic form $\lambda' \mapsto \sum_{x \in Z_{\text{red}}} \operatorname{sgn}(x)||\lambda'(x)||^2$.

For any Euclidean (1|1)-manifold (Y, ω), define a real vector space $L(Y, \omega)$ as follows. It is a subspace of $\mathcal{O}(Y)^{\text{odd}}$. An element $\psi \in \mathcal{O}(Y)^{\text{odd}}$ is in $L(Y, \omega)$ iff, for any open set $U \subset Y_{\text{red}}$ with $\omega|_{Y_U} = ds_U + \lambda_U d\lambda_U$ for some $s_U \in \mathcal{O}(Y_U)^{\text{even}}$, $\lambda_U \in \mathcal{O}(Y_U)^{\text{odd}}$, the restriction $\psi|_U$ is a constant multiple of λ_U. A map $\tau : (Y, \omega) \to (Y', \omega')$ induces a linear map $\tau^* : L(Y', \omega') \to L(Y, \omega)$. By lemma 1.1.7, (Y, ω) induces a spin structure on Y_{red}; $L(Y, \omega)$ can then be described as in [ST04], i.e. the space of sections of the positive spinor bundle S^+ satisfying the Dirac equation.

The *fermionic Fock space* of a Euclidean (1|1)-manifold (Y, ω) is

$$F(Y, \omega) = \Lambda^{\text{top}} L(Y_1, \omega|_{Y_1})^{\vee} \otimes \Lambda^*\big(L(Y_2, \omega|_{Y_2})^{\vee}\big),$$

[9] More conceptually, the operators $E_B(I_{z,\theta})$ are determined by a single one because the Lie algebra of the super semigroup described in footnote 6 is generated by a single element. [DM99]

where Y_1 is the union of the closed components of Y and $Y_2 = Y - Y_1$. The Fock space depends covariantly with (Y, ω). Let us recall a number of its properties from [ST04].

Suppose (Z_i, ξ_i), $i = 1, 2$, are two Euclidean $(0|1)$-manifolds with $(Z_i)_{\text{red}}$ oriented. A *bordism* $(Y, \omega, \alpha, \beta)$ from (Z_1, ξ_1) to (Z_2, ξ_2) consists of a Euclidean $(1|1)$-manifold (Y, ω) with Y_{red} compact, and two maps

$$\begin{tikzcd}
& (Y,\omega) & \\
(Z_1,\xi_1) \arrow[ur, "\alpha"] & & (Z_2,\xi_2), \arrow[ul, "\beta"']
\end{tikzcd}$$

such that $\alpha_{\text{red}} \cup \beta_{\text{red}}$ maps $(Z_1)_{\text{red}} \cup (Z_2)_{\text{red}}$ bijectively onto ∂Y_{red} with α_{red} orientation-reversing and β_{red} orientation-preserving. (Recall, by lemma 1.1.7, the metric induces an orientation on Y_{red}, and hence on ∂Y_{red}.) *Each bordism $(Y, \omega, \alpha, \beta)$ induces a $C(Z_2, \xi_2)$-$C(Z_1, \xi_1)$-action on $F(Y, \omega)$.* [10] This action has the following properties. Suppose (Z_3, ξ_3) is another Euclidean $(0|1)$-manifold with $(Z_3)_{\text{red}}$ oriented and $\mu : (Z_3, \xi_3) \to (Z_2, \xi_2)$ is an invertible, orientation-preserving map. The $C(Z_3, \xi_3)$-$C(Z_1, \xi_1)$-action on $F(Y, \omega)$ induced by the new bordism $(Y, \omega, \alpha, \beta\mu)$ pulls back to the above $C(Z_2, \xi_2)$-$C(Z_1, \xi_1)$-action along the algebra isomorphism $\mu^* : C(Z_2, \xi_2) \to C(Z_3, \xi_3)$. On the other hand, any invertible map $\tau : (Y, \omega) \to (Y', \omega')$ defines another bordism $(Y', \omega', \tau\alpha, \tau\beta)$, as well as an isomorphism $\tau_* : F(Y, \omega) \to F(Y', \omega')$ of the induced $C(Z_2, \xi_2)$-$C(Z_1, \xi_1)$-bimodules.

Composition of bordisms gives rise to tensor product of bimodules. More precisely, suppose we have a diagram

$$\begin{tikzcd}
& & (Y'',\omega'') & & \\
& (Y,\omega) \arrow[ur, "\iota"] & & (Y',\omega') \arrow[ul, "\iota'"'] & \\
(Z_1,\xi_1) \arrow[ur, "\alpha"] & & (Z_2,\xi_2) \arrow[ul, "\beta"'] \arrow[ur, "\alpha'"] & & (Z_3,\xi_3), \arrow[ul, "\beta'"']
\end{tikzcd}$$

where $(Y, \omega, \alpha, \beta)$ and $(Y', \omega', \alpha', \beta')$ are bordisms and the square is a pushout. Then $(Y'', \omega'', \iota\alpha, \iota'\beta')$ is also a bordism, and there is a canonical isomorphism

$$F(Y', \omega') \otimes_{C(Z_2,\xi_2)} F(Y, \omega) \cong F(Y'', \omega'') \tag{1.1.26}$$

of $C(Z_3, \xi_3)$-$C(Z_1, \xi_1)$-bimodules.

For each $p, q \geq 0$, let $\lambda \in \mathcal{O}(Z_{p,q})^{\text{odd}}$ be the canonical element defined earlier. Notice that

$$C(Z_{p,q}) := C(Z_{p,q}, \lambda d\lambda) \cong Cl_{p,q}.$$

If $\epsilon : (Z_{p,q}, \lambda d\lambda) \to (Z_{p,q}, \lambda d\lambda)$ is given by $\lambda \mapsto -\lambda$, the induced map ϵ^* on $C(Z_{p,q})$ is precisely the grading homomorphism. For $t > 0$, we use the notations

$$F(\mathbb{R}^{1|1}_{[0,t]}) := F(\mathbb{R}^{1|1}_{[0,t]}, ds + \lambda d\lambda),$$
$$\Omega := 1 \in \Lambda^0\big(L(\mathbb{R}^{1|1}_{[0,t]}, ds + \lambda d\lambda)^\vee\big) \subset F(\mathbb{R}^{1|1}_{[0,t]}).$$

[10] This action is an example of the fermionic Fock representation.

The element Ω has the following properties. The self-maps of $(\mathbb{R}^{1|1}_{[0,t]}, ds + \lambda d\lambda)$, which consist of $\tau_0 = 1$ and $\tau_0\epsilon = \epsilon$ in (1.1.5), preserve Ω. For any appropriate bordism that induces the structure of a $C(Z_{1,0})$-$C(Z_{1,0})$-bimodule on $F(\mathbb{R}^{1|1}_{[0,t]})$, the bimodule is generated by Ω (hence irreducible) and has the property that $\lambda\Omega = \Omega\lambda$, where $\lambda \in \mathcal{O}(Z_{1,0})^{\mathrm{odd}} \subset C(Z_{1,0})$. Also, any isomorphism $F(\mathbb{R}^{1|1}_{[0,s]}) \otimes_{C(Z_{1,0})} F(\mathbb{R}^{1|1}_{[0,t]}) \cong F(\mathbb{R}^{1|1}_{[0,s+t]})$ corresponding to gluing bordisms sends $\Omega \otimes \Omega$ to Ω.

Lemma 1.1.27. *Suppose (M_i, χ_i), $i = 1, 2$, are Euclidean $(0|1)$- or $(1|1)$-manifolds and B is a super manifold with $B_{red} =$ a point. There is a map*

$$\ell : Hom\big(M_1(B), (\chi_1)_B; M_2(B), (\chi_2)_B\big) \to Hom(M_1, \chi_1; M_2, \chi_2),$$

such that $\ell(\phi)_{red} = \phi_{red}$ (see lemma 1.1.15) and it is a left inverse of (1.1.14). Also, $\ell(\phi'\phi) = \ell(\phi')\ell(\phi)$ whenever ϕ and ϕ' are composable.

Proof. See appendix A. □

This technical result allows us to replace bordisms with B-bordisms in the above discussion of the Clifford algebras $C(Z, \xi)$ and the Fock spaces $F(Y, \omega)$. [11] For any B-isomorphism μ, say on $Z_{p,q}$, the map $\ell(\mu)$ induces an automorphism $\ell(\mu)^*$ on $C(Z_{p,q})$. Given a B-bordism $(Y, \omega, \alpha, \beta)$, it follows from $\ell(\alpha)_{\mathrm{red}} = \alpha_{\mathrm{red}}$ and $\ell(\beta)_{\mathrm{red}} = \beta_{\mathrm{red}}$ that $(Y, \omega, \ell(\alpha), \ell(\beta))$ is a bordism, which induces a bimodule structure on $F(Y, \omega)$. That ℓ respects compositions implies that an equivalence of B-bordisms yields an isomorphism of bimodules. Similarly, composition of B-bordisms gives rise to composition of bordisms, and hence tensor product of bimodules.

Now, we define a super category $\mathcal{SEB}^1_n$ for each $n \in \mathbb{Z}$. Let us first fix an object B of $\mathcal{S}$ and describe the category $\mathcal{SEB}^1_n(B)$. The objects of $\mathcal{SEB}^1_n(B)$ are $Z_{p,q}$, $p, q \geq 0$. There are two types of morphisms, 'decorated B-isomorphisms' and equivalence classes of 'decorated B-bordisms.' A *decorated B-isomorphism* (μ, c) consists of a B-isomorphism μ from some $Z_{p,q}$ to itself, and an element $c \in C(Z_{p,q})^{\otimes -n}$. [12] One should think of c as coming from the first $Z_{p,q}$. A *decorated B-bordism* is a 5-tuple $(Y, \omega, \alpha, \beta, \Psi)$, where the first four components comprise a B-bordism and $\Psi \in F(Y, \omega)^{\otimes -n}$. This 5-tuple is *equivalent* to precisely those of the form $\big(Y', \omega', \tau\alpha, \tau\beta, \ell(\tau)_*\Psi\big)$, for some invertible map $\tau : (Y(B), \omega_B) \to (Y'(B), \omega'_B)$. Denote the equivalence class by $[Y, \omega, \alpha, \beta, \Psi]$. Composition in $\mathcal{SEB}^1_n(B)$ is defined as follows:

$$\begin{array}{lcl}
(\mu', c') \circ (\mu, c) & = & (\mu'\mu, \ell(\mu)^* c' \cdot c), \\
(\mu', c') \circ [Y, \omega, \alpha, \beta, \Psi] & = & [Y, \omega, \alpha, \beta\mu'^{-1}, (\mu'^{-1})^* c'\Psi], \\
[Y', \omega', \alpha', \beta', \Psi'] \circ (\mu, c) & = & [Y', \omega', \alpha'\mu, \beta', \Psi' c], \\
[Y', \omega', \alpha', \beta', \Psi'] \circ [Y, \omega, \alpha, \beta, \Psi] & = & [Y'', \omega'', \iota\alpha, \iota'\beta', \Psi''].
\end{array}$$

For the last case, $(Y'', \omega'', \iota\alpha, \iota'\beta')$ is as in (1.1.18), and Ψ'' is the image of $\Psi' \otimes \Psi$ under the induced isomorphism (1.1.26). Also, if a decorated B-bordism is written as $(Y, \omega, \alpha, \beta, c_1\Psi c_2)$, where $\Psi \in F(Y, \omega)$ and c_1, c_2 are Clifford algebra elements, $c_1\Psi c_2$ is defined using the bimodule structure on $F(Y, \omega)$ induced by $(Y, \omega, \alpha, \beta)$.

For any morphism $f : B' \to B$ in $\mathcal{S}$, the functor $\mathcal{SEB}^1_n(f)$ is identity on objects and on decorated B-isomorphisms. As to decorated B-bordisms, we first define, for $(z, \theta) \in \mathcal{O}(B)^{\mathrm{even}} \times \mathcal{O}(B)^{\mathrm{odd}}$ and

[11] A more desirable approach would be to find appropriate functor-of-points interpretations of $C(Z, \xi)$ and $F(Y, \omega)$. However, the author has not been able to achieve that, and resorts to this ad hoc approach.

[12] If $n > 0$, we use the notation $C(Z_{p,q})^{\otimes -n}$ for the opposite algebra of $C(Z_{p,q})^{\otimes n}$ and also $F(Y, \omega)^{\otimes -n}$ for the opposite bimodule of $F(Y, \omega)^{\otimes n}$. See [ST04].

$\Psi \in F(\mathbb{R}^{1|1}_{[0,z^0]}, ds + \lambda d\lambda)$,

$$I_{z,\theta,\Psi} = [\mathbb{R}^{1|1}_{[0,z^0]}, ds + \lambda d\lambda, \gamma_{0,0}, \gamma_{z,\theta}, \Psi], \qquad I_{z,\theta} = I_{z,\theta,\Omega^{\otimes -n}}$$

and the related morphisms $\bar{I}_{z,\theta,\Psi}$, $LI_{z,\theta,\Psi}$, $RI_{z,\theta,\Psi}$ as in (1.1.21). Every equivalence class of decorated B-bordisms is a disjoint union of ones of the form $I_{z,\theta,\Psi}$, $\bar{I}_{z,\theta,\Psi}$, $LI_{z,\theta,\Psi}$, $RI_{z,\theta,\Psi}$ and $[Y, \omega, -, -, \Psi]$, where $Y_{\mathrm{red}} = S^1$. We define $\mathcal{SEB}^1_n(f)$ to send $I_{z,\theta,\Psi}$ to $I_{f^*z, f^*\theta,\Psi}$ (and similarly for $\bar{I}_{z,\theta,\Psi}$, $LI_{z,\theta,\Psi}$, $RI_{z,\theta,\Psi}$), $[Y, \omega, -, -, \Psi]$ to itself, and preserve disjoint unions. This finishes the definition of the super category $\mathcal{SEB}^1_n$.

Let us compute the relations among certain endomorphisms of $Z_{1,0}$ in $\mathcal{SEB}^1_n(B)$. For any $c \in C(Z_{1,0})^{\otimes -n}$, we have

$$(1, c) \circ (\epsilon, 1) = (\epsilon, \epsilon^* c) = (\epsilon, 1) \circ (1, \epsilon^* c). \tag{1.1.28}$$

Notice that, for the B-isomorphism ϵ, $\ell(\epsilon)$ is also denoted ϵ (A.3), which induces the grading homomorphism ϵ^* on $C(Z_{1,0})^{\otimes -n} \cong Cl_{-n}$. Since $\lambda\Omega = \Omega\lambda$, where $\lambda \in \mathcal{O}(Z_{1,0})^{\mathrm{odd}}$ generates $C(Z_{1,0})$, we have

$$(1, c) \circ I_{z,\theta} = I_{z,\theta} \circ (1, c). \tag{1.1.29}$$

The following analogues of (1.1.19) and (1.1.20),

$$(\epsilon, 1) \circ I_{z,\theta,\Psi} \circ (\epsilon, 1) = I_{z,-\theta,\Psi}, \tag{1.1.30}$$
$$I_{z_1,\theta_1} \circ I_{z_2,\theta_2} = I_{z_1+z_2+\theta_1\theta_2, \theta_1+\theta_2}, \tag{1.1.31}$$

are proved in the same way. For (1.1.31), we also need the fact that Ω is preserved in tensor products associated with composing B-bordisms.

Definition 1.1.32. A 1-*SEFT of degree* n is a functor $E : \mathcal{SEB}^1_n \to \mathcal{S}\mathrm{Hilb}$ of super categories. Let $E_B : \mathcal{SEB}^1_n(B) \to \mathcal{S}\mathrm{Hilb}(B)$ be the (ordinary) functor associated with each object B of $\mathcal{S}$. Since the values of E_B on objects and on decorated B-isomorphisms does not depend on B, [13] we simply write $E(Z_{p,q})$ and $E(\mu, c)$. Now, fix B and let $\Lambda = \mathcal{O}(B)$. The functor E_B is required to satisfy the following assumptions:

(i) E_B sends disjoint unions of objects (resp. morphisms) to tensor products of graded Hilbert spaces (resp. operators).

(ii) E_B sends objects (resp. B-bordisms) with opposite orientations to the same graded Hilbert space (resp. operator).

(iii) Given a B-bordism $(Y, \omega, \alpha, \beta)$ from $Z_{p,q}$ to $Z_{r,s}$, the map

$$\mathrm{Ad}_{E(Z_{p,q})} : \Lambda \otimes E(Z_{q,p}) \otimes E(Z_{r,s}) \to \Lambda \otimes B\big(E(Z_{p,q}), E(Z_{r,s})\big),$$

sends $E_B([Y, \omega, -, \alpha \sqcup \beta, \Psi])$ to $E_B([Y, \omega, \alpha, \beta, \Psi])$, for any $\Psi \in F(Y, \omega)$.

(iv) The operator $\epsilon = E(\epsilon, 1)$ on $E(Z_{1,0})$ is the grading homomorphism.

(v) $E_B(I_{z,\theta})$ is analytic in (z, θ).

[13] See definition 1.1.22.

(vi) $E(\mu, c)$ is linear in c; $E_B([Y, \omega, \alpha, \beta, \Psi])$ is linear in Ψ.

Remark. The two versions of degree zero 1-SEFT, i.e definition 1.1.22 and the case $n = 0$ of definition 1.1.32, are equivalent. Indeed, since $C(Z_{p,q})^{\otimes 0} = \mathbb{R} = F(Y, \omega)^{\otimes 0}$, we may regard $\mathcal{SEB}^1$ as a subcategory of $\mathcal{SEB}^1_0$ via $Z_{p,q} \mapsto Z_{p,q}$, $\mu \mapsto (\mu, 1)$ and $[Y, \omega, \alpha, \beta] \mapsto [Y, \omega, \alpha, \beta, 1]$. Clearly, any functor $\mathcal{SEB}^1_0 \to \mathcal{S}\mathrm{Hilb}$ satisfying assumption (vi) of 1.1.32 is uniquely determined by its restriction to $\mathcal{SEB}^1$, and any functor $\mathcal{SEB}^1 \to \mathcal{S}\mathrm{Hilb}$ extends linearly to $\mathcal{SEB}^1_0$.

Given a degree n 1-SEFT E, (i) and (ii) imply $E(Z_{p,q}) = \mathcal{H}^{\otimes(p+q)}$, where $\mathcal{H} = E(Z_{1,0})$. For decorated B-isomorphisms (μ, c), it suffices by (i) and (iv) to consider the cases $\mu = 1$ and $c \in C(Z_{1,0})^{\otimes -n} \cong Cl_{-n}$. The corresponding operators $E(1, c)$ define a Cl_{-n}-action on $\mathcal{H}$. By (1.1.28), it is in fact a graded Cl_{-n}-action. Now, fix an object B of $\mathcal{S}$. The same arguments following definition 1.1.22 show that the values of E_B on all decorated B-bordisms are determined by the values on $I_{z,\theta,\Psi}$. Since $\Psi \in F(\mathbb{R}^{1|1}_{[0,z^0]})^{\otimes -n}$, which is generated by $\Omega^{\otimes -n}$ as a $C(Z_{1,0})^{\otimes -n}$-$C(Z_{1,0})^{\otimes -n}$-bimodule, the operators $E_B(I_{z,\theta,\Psi})$ are in turn determined by $E_B(I_{z,\theta})$ and the Cl_{-n}-action on $\mathcal{H}$. (Recall: $I_{z,\theta} := I_{z,\theta,\Omega^{\otimes -n}}$.) The morphisms $I_{z,\theta}$ satisfy (1.1.30) and (1.1.31), hence the same analysis following definition 1.1.22 applies, again yielding (1.1.25):

$$E_B(I_{z,\theta}) = X(z) + \theta Y(z) = \begin{cases} e^{-zQ^2} + \theta Q e^{-zQ^2} & \text{on } \mathcal{H}' \\ 0 & \text{on } \mathcal{H}'^{\perp} \end{cases},$$

where Q is an odd, self-adjoint operator with compact resolvent on a graded subspace $\mathcal{H}' \subset \mathcal{H}$. Furthermore, by (1.1.29), all $E_B(I_{z,\theta})$ are Cl_{-n}-linear, hence $\mathcal{H}'$ is in fact a graded Cl_{-n}-submodule and Q is Cl_{-n}-linear. The operator Q is called *the generator of E*.

1.2. Categories $\mathcal{SEFT}_n$

Throughout this section, we fix an integer n and a real Hilbert space $\mathcal{H}$ with a stable, orthogonal [14] graded Cl_{-n}-action.

Definition 1.2.1. $\mathcal{SEFT}_n(\mathcal{H})$ is a topological category whose objects are 1-SEFTs E such that $E(Z_{1,0}) = \mathcal{H}$. We take the discrete topology on the set of objects. A morphism from E to E' is a 'deformation' from the spectral decomposition of the generator Q_E of E to the spectral decomposition of the generator $Q_{E'}$ of E'. To explain what this means, let us first describe the spectral decomposition of the generator of a 1-SEFT of degree n.

Let E be a 1-SEFT. Recall that its generator Q_E is an (i) odd, (ii) self-adjoint, (iii) Cl_{-n}-linear operator (defined on a graded Cl_{-n}-submodule of $\mathcal{H}$) (iv) with compact resolvent. Because of (ii), Q_E has real eigenvalues and its domain decomposes into eigenspaces. (The orthogonal complement of the domain of Q_E, if nontrivial, should be regarded as the eigenspace associated with the eigenvalue ∞.) It is useful to have in mind the picture in figure 1. The line represents the real numbers. The points on the real line are the eigenvalues of Q_E and they are labeled by the corresponding eigenspaces. By (iii), the eigenspaces are Cl_{-n}-submodules. Then, (iv) implies that the eigenvalues have no accumulation point on the real line and each eigenspace is finite dimensional. Finally, (i) has the following implication. If ϵ denotes the grading homomorphism on $\mathcal{H}$, (i) means $\epsilon Q_E = -Q_E \epsilon$, and hence ϵ maps the eigenspace V^E_λ to $V^E_{-\lambda}$. This gives

[14] A graded Cl_{-n}-action on a vector space is *stable* if each isomorphism class of irreducible graded Cl_{-n}-module appears with infinite muliplicity; it is *orthogonal* if $\mathcal{H}^{\mathrm{even}} \perp \mathcal{H}^{\mathrm{odd}}$ and the standard basis elements $e_1, \ldots, e_{|n|}$ of $\mathbb{R}^{|n|}$, regarded as elements of Cl_{-n}, act orthogonally on $\mathcal{H}$.

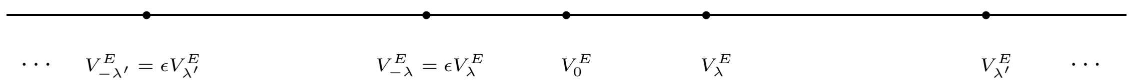

Figure 1: The spectral decomposition of Q_E for a 1-SEFT E.

rise to a symmetry about the origin in figure 1. Notice that V^E_0 is the only eigenspace that is a *graded* Cl_{-n}-module.

We now continue with the definition of $\mathcal{SEFT}_n(\mathcal{H})$. Given two objects E, E', let V^E_λ (resp. $V^{E'}_\lambda$) denote the eigenspace of Q_E (resp. $Q_{E'}$) with eigenvalue λ. A deformation (α, f, A) from the spectral decomposition of Q_E to that of $Q_{E'}$ consists of the following data:

- a map $\alpha : \operatorname{sp} Q_E \to \operatorname{sp} Q_{E'}$, which is odd, proper and order preserving,
- an even linear map $f : \bigoplus_\lambda V^E_\lambda \to \bigoplus_{\lambda'} V^{E'}_{\lambda'}$, which embeds each V^E_λ isometrically into $V^{E'}_{\alpha(\lambda)}$, and
- an (ungraded) Cl_{-n}-submodule A of $\mathcal{H}$, such that $Q_{E'}$ is nonnegative on A, $A \perp \epsilon A$ and $\bigoplus_{\lambda'} V^{E'}_{\lambda'} = f(\bigoplus_\lambda V^E_\lambda) \oplus A \oplus \epsilon A$.

In particular, fixing α,

$$f \in \prod_{\lambda \in \operatorname{sp} Q_E} \operatorname{Hom}(V^E_\lambda, V^{E'}_{\alpha(\lambda)}),$$

where $\operatorname{Hom}(\cdot,\cdot)$ denotes the space of linear maps between two finite dimensional vector spaces, with the usual topology. Furthermore, given α and f, A is determined by

$$A \cap V^{E'}_0 \in \operatorname{Gr}(V^{E'}_0),$$

where $\operatorname{Gr}(\cdot)$ denotes the space of linear subspaces in a finite dimensional vector space, again with the usual topology. The set of morphisms $\{(\alpha, f, A)\}$ from E to E' is topologized as a subspace of

$$\coprod_\alpha \left(\prod_{\lambda \in \operatorname{sp} Q_E} \operatorname{Hom}(V^E_\lambda, V^{E'}_{\alpha(\lambda)}) \times \operatorname{Gr}(V^{E'}_0) \right),$$

which has one component for each α. Composition of morphisms is defined by

$$(\alpha', f', A') \circ (\alpha, f, A) = (\alpha'\alpha, f'f, f'(A) \oplus A').$$

This finishes the definition of $\mathcal{SEFT}_n(\mathcal{H})$. Sometimes, we simply write $\mathcal{SEFT}_n$, with the underlying Hilbert space $\mathcal{H}$ unspecified but understood.

Remarks. (a) The assumptions that α is odd and f is even imply that $f|_{V^E_{-\lambda}} = \epsilon \circ f|_{V^E_\lambda} \circ \epsilon$. In this sense, the symmetry about the origin in the spectral decompositions associated with the objects is respected by the morphisms. (b) There are three special types of morphisms. (See figure 2.) For the first type, $\operatorname{sp} Q_E = \operatorname{sp} Q_{E'}$, α is the identity map and $A = 0$, which means the eigenvalues are the same but the eigenspaces change by isomorphisms. For the second type, $A = 0$ and f is identity, which means the

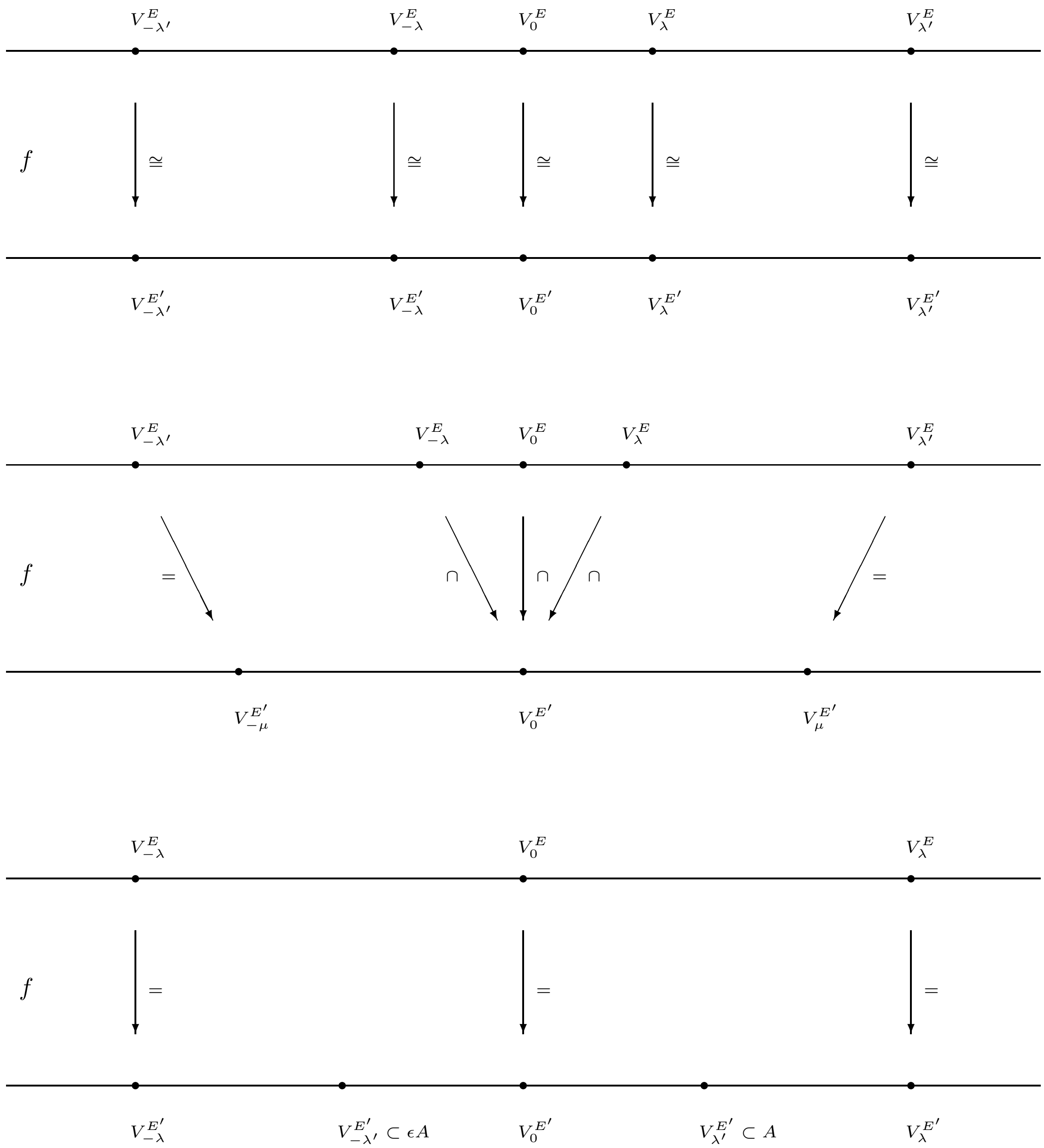

Figure 2: Three basic types of morphisms in $\mathcal{SEFT}_n$.

eigenspaces remain the same but the eigenvalues may have shifted, with the possibility of a finite number of eigenvalues merging into one and the corresponding eigenspaces combining together. For the third type, $\operatorname{sp} Q_E \subset \operatorname{sp} Q_{E'}$, α is the inclusion and f is also the inclusion, which means the original eigenvalues and eigenspaces are unchanged but there are new ones that 'emerge from $+\infty$ (those in A) and $-\infty$ (those in ϵA).' All morphisms are generated by these three special types. Indeed, any (α, f, A) can be factored as

$$(\text{inclusion}, \text{inclusion}, A) \circ (\text{identity}, f, 0) \circ (\alpha, \text{identity}, 0).$$

The result below concerns the homotopy types of $|\mathcal{SEFT}_n(\mathcal{H})|$. To state the result, we need to first define a space of operators $Fred_n(\mathcal{H})$. Let $F_n(\mathcal{H})$ denote the space (with the norm topology) of Fredholm operators on $\mathcal{H}$ which are odd, self-adjoint and Cl_{-n}-linear. If $n \not\equiv 1 \pmod 4$, we define $Fred_n(\mathcal{H}) = F_n(\mathcal{H})$. If $n \equiv 1 \pmod 4$, consider an orthonormal basis $e_1, \ldots, e_{|n|}$ of $\mathbb{R}^{|n|}$, regarded as elements of Cl_{-n} and in turn as orthogonal operators on $\mathcal{H}$. For each $F \in F_n(\mathcal{H})$, the operator $e_1 \cdots e_{|n|} F$ is even and self-adjoint. $Fred_n(\mathcal{H})$ is defined to be the subspace of $F_n(\mathcal{H})$ consisting of those F such that

$$(e_1 \cdots e_{|n|} F)|_{\mathcal{H}^{\text{even}}} \text{ is neither essentially positive nor essentially negative.} \tag{1.2.2}$$

[15] Sometimes we write $Fred_n$ instead of $Fred_n(\mathcal{H})$ when the underlying Hilbert space is understood.

Theorem 1.2.3. *For each $n \in \mathbb{Z}$, there is a homotopy equivalence*

$$|\mathcal{SEFT}_n(\mathcal{H})| \simeq Fred_n(\mathcal{H}).$$

Remark. It follows from [AS69] (see also [LM89], III, §10) that, for each $n \leq 0$, $Fred_n$ represents the functor KO^n. We claim that this is also true for $n > 0$. Suppose $n = 4k - \ell$, where $k > 0$ and $\ell \geq 0$ are integers. Let $\mathcal{H}$ continue to be a Hilbert space with a stable graded Cl_{-n}-action and V be an irreducible graded $Cl_{\ell,\ell}$-module. Since $Cl_{0,n} \otimes Cl_{\ell,\ell} \cong Cl_{\ell,4k} \cong Cl_{4k+\ell}$ as a graded tensor product, $\mathcal{H} \otimes V$ admits a stable graded $Cl_{4k+\ell}$-action with the even and odd parts being $(\mathcal{H}^{\text{even}} \otimes V^{\text{even}}) \oplus (\mathcal{H}^{\text{odd}} \otimes V^{\text{odd}})$ and $(\mathcal{H}^{\text{even}} \otimes V^{\text{odd}}) \oplus (\mathcal{H}^{\text{odd}} \otimes V^{\text{even}})$. Let us express every operator on a graded vector space as a 2×2 block with respect to the parity decomposition of the vector space. The map

$$\begin{pmatrix} 0 & F^1 \\ F^0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -F^0 \otimes 1_{V^{\text{odd}}} + F^1 \otimes 1_{V^{\text{even}}} \\ F^0 \otimes 1_{V^{\text{even}}} - F^1 \otimes 1_{V^{\text{odd}}} & 0 \end{pmatrix} \tag{1.2.4}$$

defines a homeomorphism from $Fred_n(\mathcal{H})$ to $Fred_{-4k-\ell}(\mathcal{H} \otimes V)$. [16] Since $n \equiv -4k - \ell \pmod 8$, this shows that $Fred_n(\mathcal{H})$ represents $KO^{-4k-\ell} = KO^n$.

By the above remark, theorem 1.2.3 implies that the space $|\mathcal{SEFT}_n|$ represents KO^n. The proof of the theorem, which is given in section 1.4, consists of two steps

$$|\mathcal{SEFT}_n(\mathcal{H})| \xrightarrow{\sim} |\mathcal{V}_n(\mathcal{H})| \simeq Fred_n(\mathcal{H}),$$

where $\mathcal{V}_n(\mathcal{H})$ is a category we define below.

[15] A self-adjoint operator is essentially positive (resp. negative) if it is positive (resp. negative) on an invariant subspace of a finite codimension.

[16] Had we chosen to use skew-adjoint, Clifford anitlinear operators to define $Fred_n$, the map in (1.2.4) would simply be $F \mapsto F \otimes \mathrm{id}_V$. (See [AS69], Theorem 5.1.)

1.3. Categories $\mathcal{V}_n$

Definition 1.3.1. Let n be an integer and $\mathcal{H}$ a real Hilbert space with a stable, orthogonal (see footnote 14) graded Cl_{-n}-action. $\mathcal{V}_n(\mathcal{H})$ is a topological category whose objects are finite dimensional graded Cl_{-n}-submodules of $\mathcal{H}$, and the set of objects is discrete. A morphism from an object V to another object V' is a pair (f, A), where $f : V \hookrightarrow V'$ is an injective map of graded Cl_{-n}-modules, and $A \subset V'$ is an (ungraded) Cl_{-n}-submodule such that $A \perp \epsilon A$, $A \perp f(V)$ and $V' = f(V) \oplus A \oplus \epsilon A$. The set of morphisms from V to V' is topologized as a subspace of $\mathrm{Hom}(V, V') \times \mathrm{Gr}(V')$. Composition is defined by

$$(f', A') \circ (f, A) = (f'f, f'(A) \oplus A').$$

For the proof of theorem 1.2.3, the reader can go directly to section 1.4. The rest of this section is a study of the categories $\mathcal{V}_n(\mathcal{H})$.

Comparing with Quillen's categories $\widehat{Vect}$ and $QVect$. Each $\mathcal{V}_n$ is designed to admit a functor from $\mathcal{SEFT}_n$, as we will see in the next section. Curiously, $\mathcal{V}_0$ and $\mathcal{V}_1$ coincide with two categories studied by Quillen [Qui73] and Segal [Seg77].

Let us first recall $\widehat{Vect}$, the category of 'virtual vector spaces.' To make explicit the dependence on two underlying infinite dimensional Hilbert spaces $\mathcal{H}^0$ and $\mathcal{H}^1$, we use the notation $\widehat{Vect}(\mathcal{H}^0, \mathcal{H}^1)$. An object of $\widehat{Vect}(\mathcal{H}^0, \mathcal{H}^1)$ is a pair (V^0, V^1), where $V^0 \subset \mathcal{H}^0$ and $V^1 \subset \mathcal{H}^1$ are finite dimensional subspaces. A morphism from an object (V^0, V^1) to another object (V'^0, V'^1) is a triple $(f^0, f^1; \phi)$, where $f^i : V^i \to V'^i$ are isometric injections and $\phi : U^0 \xrightarrow{\sim} U^1$ is an isometric isomorphism, U^i being the orthogonal complement of $f^i(V^i)$ in V'^i, $i = 0, 1$. The composition of $(f^0, f^1; \phi)$ with a morphism $(f'^0, f'^1; \phi')$ from (V'^0, V'^1) to (V''^0, V''^1) is defined by

$$(f'^0, f'^1; \phi') \circ (f^0, f^1; \phi) = (f'^0 f^0, f'^1 f^1; f'^1 \phi (f'^0)^{-1} \oplus \phi').$$

More precisely, the map $(f'^0)^{-1}$ appearing here is defined on $f'^0(U^0)$, so that $f'^1 \phi f'^{0^{-1}} \oplus \phi'$ is indeed defined on the orthogonal complement of $(f'^0 f^0)(V^0)$ in V''^0.

Let $Vect$ denote the category of finite dimensional vector spaces (say in $\mathcal{H}^0$) and isomorphisms. Taking direct sums defines a symmetric monoidal structure on $Vect$. It is known [Seg77] that

$$|\widehat{Vect}(\mathcal{H}^0, \mathcal{H}^1)| \simeq \Omega B|Vect| \simeq \Omega B \left(\coprod_{n \geq 0} BGL_n \right),$$

i.e. $\widehat{Vect}(\mathcal{H}^0, \mathcal{H}^1)$ is a 'category-level group completion' of $Vect$.

Suppose $\mathcal{H}$ is a graded Hilbert space such that $\mathcal{H}^{\mathrm{even}} = \mathcal{H}^0$, $\mathcal{H}^{\mathrm{odd}} = \mathcal{H}^1$ are both infinite dimensional. We have the following

Proposition 1.3.2. *The categories $\mathcal{V}_0(\mathcal{H})$ and $\widehat{Vect}(\mathcal{H}^0, \mathcal{H}^1)$ are isomorphic.*

Proof. We define a functor

$$F : \mathcal{V}_0(\mathcal{H}) \to \widehat{Vect}(\mathcal{H}^0, \mathcal{H}^1).$$

For an object V of $\mathcal{V}_0(\mathcal{H})$, i.e. a finite dimensional graded subspace of $\mathcal{H}$, we have

$$F(V) = (V \cap \mathcal{H}^0, V \cap \mathcal{H}^1).$$

Suppose (f, A) is a morphism from V to V' in $\mathcal{V}_0(\mathcal{H})$. By definition, the subspaces A, ϵA, and $f(V)$ of V' are pairwise orthogonal such that $V' = f(V) \oplus A \oplus \epsilon A$. Define a map $\phi_A : (A \oplus \epsilon A) \cap \mathcal{H}^0 \to (A \oplus \epsilon A) \cap \mathcal{H}^1$ as follows. Let $u = a + \epsilon a'$ be an element of $(A \oplus \epsilon A) \cap \mathcal{H}^0$, where $a, a' \in A$. Since $\epsilon u = u$, we in fact have $a = a'$, hence $(A \oplus \epsilon A) \cap \mathcal{H}^0$ consists precisely of elements of the form $a + \epsilon a$, $a \in A$. Similarly, $(A \oplus \epsilon A) \cap \mathcal{H}^1$ consists precisely of elements of the form $a - \epsilon a$, $a \in A$. The map ϕ_A is defined by $\phi_A(a + \epsilon a) = a - \epsilon a$ for any $a \in A$. Now, the definition of the functor F on morphisms is given by

$$F(f, A) = (f|_{V \cap \mathcal{H}^0}, f|_{V \cap \mathcal{H}^1}; \phi_A).$$

Indeed, $f|_{V \cap \mathcal{H}^i}$ embeds $V \cap \mathcal{H}^i$ isometrically into $V' \cap \mathcal{H}^i$, and $(A \oplus \epsilon A) \cap \mathcal{H}^i$ is the orthogonal complement of $f(V \cap \mathcal{H}^i)$ in $V' \cap \mathcal{H}^i$, where $i = 0, 1$. Also, since $A \perp \epsilon A$, we have $||a + \epsilon a|| = ||a - \epsilon a||$, so that ϕ_A is indeed an isometry. It is straightforward to verify that F respects compositions.

To show that F is an isomorphism of categories, we define a functor

$$G : \widehat{Vect}(\mathcal{H}^0, \mathcal{H}^1) \to \mathcal{V}_0(\mathcal{H})$$

and show that it is the inverse of F. For an object (V^0, V^1) of $\widehat{Vect}(\mathcal{H}^0, \mathcal{H}^1)$, we have

$$G(V^0, V^1) = V^0 \oplus V^1,$$

regarded as a graded subspace of $\mathcal{H}$, hence an object of $\mathcal{V}_0(\mathcal{H})$. Suppose $(f^0, f^1; \phi)$ is a morphism from (V^0, V^1) to (V'^0, V'^1) in $\widehat{Vect}(\mathcal{H}^0, \mathcal{H}^1)$. In particular, $\phi : U^0 \to U^1$ is an isometric isomorphism, where U^i is the orthogonal complement of $f^i(V^i)$ in V'^i, $i = 0, 1$. Let $A_\phi = \{u + \phi(u) | u \in U^0\}$, and we define

$$G(f^0, f^1; \phi) = (f^0 \oplus f^1, A_\phi).$$

Notice that $\epsilon A_\phi = \{u - \phi(u) | u \in U^0\}$ and $U^0 \oplus U^1 = A_\phi \oplus \epsilon A_\phi$. Also, the facts that ϕ is an isometry and $U^0 \perp U^1$ imply $A_\phi \perp \epsilon A_\phi$. These show that $G(f^0, f^1; \phi)$ just defined is indeed a morphism from $G(V^0, V^1)$ to $G(V'^0, V'^1)$ in $\mathcal{V}_0(\mathcal{H})$. Again, it is straightforward to verify that G respects compositions.

Let us check that GF and FG are identity functors. It is clear on the objects. Suppose (f, A) is a morphism from an object V to another object V' in $\mathcal{V}_0(\mathcal{H})$. We have $GF(f, A) = (f, A_{\phi_A})$, where A_{ϕ_A} consists precisely of the elements of the form $(a + \epsilon a) + \phi_A(a + \epsilon a)$, $a \in A$. Since $\phi_A(a + \epsilon a) = a - \epsilon a$, the elements of A_{ϕ_A} are in fact of the form $2a$, where $a \in A$. Hence, $A_{\phi_A} = A$ and GF is indeed the identity on $\mathcal{V}_0(\mathcal{H})$. On the other hand, consider a morphism $(f^0, f^1; \phi)$ from an object (V^0, V^1) to another object (V'^0, V'^1) in $\widehat{Vect}(\mathcal{H}^0, \mathcal{H}^1)$. Let U^i again denote the orthogonal complement of $f^i(V^i)$ in V'^i, $i = 0, 1$. We have $FG(f^0, f^1; \phi) = (f^0, f^1; \phi_{A_\phi})$. Since $A_\phi = \{u + \phi(u) | u \in U^0\}$, the map $\phi_{A_\phi} : U^0 \to U^1$ is given by

$$\begin{aligned}
\phi_{A_\phi}(u) &= \phi_{A_\phi}\left[\left(\frac{u}{2} + \frac{\phi(u)}{2}\right) + \epsilon\left(\frac{u}{2} + \frac{\phi(u)}{2}\right)\right] \\
&= \left(\frac{u}{2} + \frac{\phi(u)}{2}\right) - \epsilon\left(\frac{u}{2} + \frac{\phi(u)}{2}\right) \\
&= \phi(u).
\end{aligned}$$

Thus, $\phi_{A_\phi} = \phi$ and FG is indeed the identity on $\widehat{Vect}(\mathcal{H}^0, \mathcal{H}_1)$. $\square$

Now we recall $QVect$, the Q-construction of $Vect$. The make explicit its dependence on an underlying infinite dimensional Hilbert space $\mathcal{H}'$, we use the notation $QVect(\mathcal{H}')$. An object of $QVect(\mathcal{H}')$ is a finite

dimensional subspace of $\mathcal{H}'$. A morphism from an object U to another object U' is a triple $(g; W_1, W_2)$, where $g : U \to U'$ is an isometric embedding, and $W_1, W_2 \subset U'$ are subspaces such that $U' = g(U) \oplus W_1 \oplus W_2$, and $g(U)$, W_1 and W_2 are pairwise orthogonal. The composition of $(g; W_1, W_2)$ with a morphism $(g'; W_1', W_2')$ from U' to U'' is given by

$$(g'; W_1', W_2') \circ (g; W_1, W_2) = (g'g; g'(W_1) \oplus W_1', g'(W_2) \oplus W_2').$$

It is known [Seg77] that

$$\Omega|QVect(\mathcal{H}')| \simeq \Omega B|Vect| \simeq \Omega B\left(\coprod_{n\geq 0} BGL_n\right).$$

Suppose $\mathcal{H}$ is a Hilbert space with a stable, orthogonal graded Cl_{-1}-action. Let $p \in Cl_{-1} = \mathbb{R} \oplus \mathbb{R}$ denote the projection onto the first factor. Notice that, as operators on a graded Cl_{-1}-module, we have $\epsilon p = (1-p)\epsilon$. We have the following

Proposition 1.3.3. *The categories $\mathcal{V}_1(\mathcal{H})$ and $QVect(p\mathcal{H})$ are isomorphic.*

Proof. We first define a functor

$$F : \mathcal{V}_1(\mathcal{H}) \to QVect(p\mathcal{H}).$$

For an object V of $\mathcal{V}_1(\mathcal{H})$, i.e. a finite dimensional Cl_{-1}-submodule of $\mathcal{H}$, we have

$$F(V) = pV.$$

Given a morphism (f, A) from an object V to another object V', we define

$$F(f, A) = (f|_{pV}; pA, p\epsilon A).$$

Here, since f embeds V isometrically into V' and f is Cl_{-1}-linear, $f|_{pV}$ embeds pV isometrically into pV'. Also, we have $pV' = f(pV) \oplus pA \oplus p\epsilon A$, hence $F(f, A)$ defined above is indeed a morphism from $F(V)$ to $F(V')$ in $QVect(p\mathcal{H})$.

Then we define a functor

$$G : QVect(p\mathcal{H}) \to \mathcal{V}_1(\mathcal{H})$$

that is the inverse of F. Given an object U of $QVect(p\mathcal{H})$, let

$$G(U) = U \oplus \epsilon U.$$

Notice that $\epsilon U = \epsilon p U = (1-p)\epsilon U \subset (1-p)\mathcal{H}$ is orthogonal to $U \subset p\mathcal{H}$, and $U \oplus \epsilon U$ is indeed a graded Cl_{-1}-submodule. Given a morphism $(g; W_1, W_2)$ from U to U' in $QVect(p\mathcal{H})$, define

$$G(g; W_1, W_2) = (g \oplus \epsilon g \epsilon, W_1 \oplus \epsilon W_2).$$

To see that the right hand side is indeed a morphism from $U \oplus \epsilon U$ to $U' \oplus \epsilon U'$ in $\mathcal{V}_1(\mathcal{H})$, observe that

$$\begin{aligned} U' \oplus \epsilon U' &= \big(g(U) \oplus W_1 \oplus W_2\big) \oplus \epsilon\big(g(U) \oplus W_1 \oplus W_2\big) \\ &= (g \oplus \epsilon g \epsilon)(U \oplus \epsilon U) \oplus (W_1 \oplus \epsilon W_2) \oplus \epsilon(W_1 \oplus \epsilon W_2). \end{aligned}$$

It is clear that $g \oplus \epsilon g \epsilon$ commutes with p and ϵ, hence is graded and Cl_{-1}-linear; also, $p(W_1 \oplus \epsilon W_2) \subset (W_1 \oplus \epsilon W_2)$, so that $W_1 \oplus \epsilon W_2$ is indeed a Cl_{-1}-submodule.

Finally, we check that GF and FG are identity functors. For objects, we observe

$$\begin{aligned} GF(V) &= pV \oplus \epsilon pV = pV \oplus (1-p)\epsilon V = pV \oplus (1-p)V = V \\ FG(U) &= p(U \oplus \epsilon U) = pU \oplus \epsilon(1-p)U = U \oplus 0 = U. \end{aligned}$$

For a morphism (f, A) from V to V' in $\mathcal{V}_1(\mathcal{H})$, we have by definition $GF(f, A) = \big(f|_{pV} \oplus \epsilon(f|_{pV})\epsilon, pA \oplus \epsilon p \epsilon A\big)$. It follows from $\epsilon p = (1-p)\epsilon$ that this is equal to (f, A). Therefore, GF is the identity on $\mathcal{V}_1(\mathcal{H})$. For a morphism $(g; W_1, W_2)$ from an object U to another object U' in $QVect(p\mathcal{H})$, we have

$$FG(g; W_1, W_2) = \big((g \oplus \epsilon g \epsilon)|_{p(U \oplus \epsilon U)}; p(W_1 \oplus \epsilon W_2), p\epsilon(W_1 \oplus \epsilon W_2)\big).$$

Since $p(U \oplus \epsilon U) = U$, $p(W_1 \oplus \epsilon W_2) = W_1$ and $p(\epsilon W_1 \oplus W_2) = W_2$, the right hand side above equals $(g; W_1, W_2)$, proving that FG is the identity on $QVect(p\mathcal{H})$. $\square$

Components of $\mathcal{V}_n$. For any $n \in \mathbb{Z}$, let $\widehat{\mathfrak{M}}_n$ denote the set of isomorphism classes of finite dimensional graded Cl_n-modules. Given such a module V, we denote its isomorphism class by $[V]$. Taking direct sums of graded Cl_n-modules induces the structure of an abelian monoid on $\widehat{\mathfrak{M}}_n$. Recall that (for $n \neq 0$) Cl_n is generated by a set of elements $e_1, \ldots, e_{|n|}$ satisfying the relations $e_i^2 = -\mathrm{sgn}(n)$ for any i, and $e_i e_j = -e_j e_i$ for any $i \neq j$.

For each n, we define a monoid homomorphism

$$i_n : \widehat{\mathfrak{M}}_n \to \widehat{\mathfrak{M}}_{n-1}.$$

For $n \geq 1$, i_n is induced by restricting a graded Cl_n-action to the action generated by $e_1, \ldots, e_{n-1}$. For $n \leq 0$, let $k = -n$ and $\eta \in \widehat{\mathfrak{M}}_{-k}$. Suppose V^e and V^o are two graded Cl_{-k}-modules representing η, and $\gamma : V^e \to V^o$ is a module isomorphism. Denote by ϵ the grading homomorphisms on both V^e and V^o. Let $V = V^e \oplus V^o$ be the graded vector space whose even and odd parts are V^e and V^o respectively. The following operators

$$e'_i = \begin{cases} \epsilon e_i \gamma & \text{on } V^e \\ -\epsilon e_i \gamma^{-1} & \text{on } V^o \end{cases} \quad 1 \leq i \leq k, \qquad e'_{k+1} = \begin{cases} \gamma & \text{on } V^e \\ \gamma^{-1} & \text{on } V^o \end{cases} \tag{1.3.4}$$

reverse the parity on V, and satisfy ${e'_i}^2 = 1$ for $1 \leq i \leq k+1$, $e'_i e'_j = -e'_j e'_i$ for $1 \leq i < j \leq k+1$. Hence, they generate a graded Cl_{-k-1}-action on V. We define $i_{-k}\eta$ to be $[V] \in \widehat{\mathfrak{M}}_{-k-1}$.

Proposition 1.3.5. *For each $n \in \mathbb{Z}$, there is a bijective correspondance*

$$\pi_0 \mathcal{V}_{-n} \cong \widehat{\mathfrak{M}}_n / i_{n+1} \widehat{\mathfrak{M}}_{n+1}. \tag{1.3.6}$$

Furthermore, the right hand side above is in fact an abelian group.

Remarks. (a) If M is an abelian monoid and $N \subset M$ is a submonoid, the set M/N is defined to be $M/\sim$, where $m \sim m'$ if and only if $m + n_1 = m' + n_2$ for some $n_1, n_2 \in N$. (b) Together with subsequent results on the categories $\mathcal{V}_{-n}$, this proposition recovers the well known interpretation of $KO^{-n}(\mathrm{pt})$ in terms of Clifford modules [ABS64], at least for $n \geq 0$. The construction of the right hand side of (1.3.6) for $n < 0$ is new to the author.

Proof of Proposition 1.3.5. Recall that the objects of $\mathcal{V}_{-n}$ are finite dimensional graded Cl_n-submodules of a stable graded Cl_n-module $\mathcal{H}$. Since submodules of every isomorphism class appear in $\mathcal{H}$ and two isomorphic ones clearly lie in the same component of $\mathcal{V}_{-n}$, there is a surjective map

$$\phi : \widehat{\mathfrak{M}}_n \to \pi_0 \mathcal{V}_{-n}.$$

Now, we show that ϕ factors through the projection $\widehat{\mathfrak{M}}_n \to \widehat{\mathfrak{M}}_n / i_{n+1}\widehat{\mathfrak{M}}_{n+1}$. It suffices to show that, for any $\eta \in \widehat{\mathfrak{M}}_n$ and $\xi \in i_{n+1}\widehat{\mathfrak{M}}_{n+1}$, $\phi(\eta) = \phi(\eta + \xi)$. Suppose $V, W \subset \mathcal{H}$ are two orthogonal graded Cl_n-submodules such that $[V] = \eta$, $[W] = \xi$. We show that there is a morphism from V to $V \oplus W$ in $\mathcal{V}_{-n}$.

First consider the case $n \geq 0$. The assumption that $[W] \in i_{n+1}\widehat{\mathfrak{M}}_{n+1}$ implies there is an operator e_{n+1} on W extending the graded Cl_n-action on W to a graded Cl_{n+1}-action. Let $A = \{w + e_{n+1}w | w \in W^{\text{even}}\} \subset W$. Since for any $w \in W^{\text{even}}$ and $1 \leq i \leq n$, we have $e_i(w + e_{n+1}w) = (e_i e_{n+1} w) + e_{n+1}(e_i e_{n+1} w)$ and $e_i e_{n+1} w \in W^{\text{even}}$, A is a Cl_n-submodule. Also, $\epsilon A = \{w - e_{n+1}w | w \in W^{\text{even}}\}$ is orthogonal to A, and the fact that e_{n+1} maps W^{even} isomorphically to W^{odd} implies $W = A \oplus \epsilon A$. Therefore, (inc, A) is a morphism from V to $V \oplus W$ in the category $\mathcal{V}_{-n}$, where inc is the inclusion of V into $V \oplus W$.

Then we consider the case $n \leq -1$. Let $k = -n$. By definition of i_{-k+1}, there are isomorphic graded $Cl_{-(k-1)}$-actions on W^{even} and W^{odd}, whose relation with the graded Cl_{-k}-action on W is described in (1.3.4). Let us use the same notations as in (1.3.4). Let $A = \{w + \epsilon\gamma w | w \in W^{\text{even}}\} \subset W$. For $1 \leq i \leq k-1$ and $w \in W^{\text{even}}$, we have

$$e_i'(w + \epsilon\gamma w) = \epsilon e_i \gamma w - \epsilon e_i \gamma^{-1} \epsilon\gamma w = (e_i w) + \epsilon\gamma(e_i w),$$

and

$$e_k'(w + \epsilon\gamma w) = \gamma w + \gamma^{-1}\epsilon\gamma w = (\epsilon w) + \epsilon\gamma(\epsilon w),$$

showing that A is a Cl_{-k} submodule of W. Since $\epsilon' A = \{w - \epsilon\gamma w | w \in W^0\}$ is orthogonal to A, (inc, A) is a morphism from V to $V \oplus W$ in $\mathcal{V}_k$, where inc is again the inclusion of V into $V \oplus W$. This finishes the proof that ϕ induces a map

$$\bar{\phi} : \widehat{\mathfrak{M}}_n / i_{n+1}\widehat{\mathfrak{M}}_{n+1} \to \pi_0 \mathcal{V}_{-n}.$$

It remains to show that $\bar{\phi}$ is one to one. Consider a morphism (f, A) from V to V' in $\mathcal{V}_{-n}$. Since $V' = f(V) \oplus A \oplus \epsilon A$ and f is a graded, Cl_n-linear embedding, we have $[V'] = [V] + [A \oplus \epsilon A] \in \widehat{\mathfrak{M}}_n$. Therefore, it suffices to show that

$$[A \oplus \epsilon A] \in i_{n+1}\widehat{\mathfrak{M}}_{n+1}. \tag{1.3.7}$$

If $n \geq 0$, the operator e_{n+1} on $A \oplus \epsilon A$ defined by $e_{n+1}|_A = -\epsilon$, $e_{n+1}|_{\epsilon A} = \epsilon$ extends the graded Cl_n-action on $A \oplus \epsilon A$ to a graded Cl_{n+1}-action, proving (1.3.7) in this case. For $n \leq -1$, let $k = -n$ and $W = A \oplus \epsilon A$. As in the proof of proposition 1.3.2, $W^{\text{even}} = \{a + \epsilon a | a \in A\}$ and $W^{\text{odd}} = \{a - \epsilon a | a \in A\}$. Let $\phi_A : W^{\text{even}} \to W^{\text{odd}}$ be the map defined by $\phi_A(a + \epsilon a) = a - \epsilon a$. We denote by $e_1', \ldots, e_k'$ the generators (as above) of the Cl_{-k}-action on W. Notice that for $i = 1, \ldots, k$, $e_i'\phi_A = \phi_A^{-1} e_i'$ as operators on W^{even}. It then follows that the following operators on W^{even}

$$\epsilon = \phi_A^{-1} e_k', \qquad e_i = \phi_A^{-1} e_i', \quad i = 1, \ldots, k-1$$

define a graded $Cl_{-(k-1)}$-action. Similarly, the following operators on W^{odd}

$$\epsilon = \phi_A e_k', \qquad e_i = -\phi_A e_i', \quad i = 1, \ldots, k-1$$

also define a graded $Cl_{-(k-1)}$-action. Furthermore, $\gamma = e_k'|_{W^{\mathrm{even}}} : W^{\mathrm{even}} \to W^{\mathrm{odd}}$ commutes with these actions. According to (1.3.4), one may use these data to define a graded Cl_{-k}-action on $W^{\mathrm{even}} \oplus W^{\mathrm{odd}} = W$, which is easily checked to coincide with the graded Cl_{-k}-action we started with. This proves $[W] \in i_{-k+1}\widehat{\mathfrak{M}}_{-k+1}$. We have now established the bijection $\pi_0 \mathcal{V}_{-n} \cong \widehat{\mathfrak{M}}_n / i_{n+1}\widehat{\mathfrak{M}}_{n+1}$.

Finally, we show that the monoid structure on $\widehat{\mathfrak{M}}_n / i_{n+1}\widehat{\mathfrak{M}}_{n+1}$ induced from taking direct sums admits an inverse. Given $\eta \in \widehat{\mathfrak{M}}_n$, let V be a graded Cl_n-module representing η. Reversing the parity in V yields another graded Cl_n-module $\bar{V}$. If ϵ denotes the grading homomorphism of V, the grading homomorphism of $V \oplus \bar{V}$ is given by $\epsilon \oplus -\epsilon$. Let $\Delta = \{(v,v) | v \in V\} \subset V \oplus \bar{V}$. Since $V \oplus \bar{V} = \Delta \oplus (\epsilon \oplus -\epsilon)\Delta$ and Δ is a Cl_n-submodule, by (1.3.7), $[V] + [\bar{V}] \in i_{n+1}\widehat{\mathfrak{M}}_{n+1}$. Therefore, the images of $[V] = \eta$ and $[\bar{V}]$ in $\widehat{\mathfrak{M}}_n / i_{n+1}\widehat{\mathfrak{M}}_{n+1}$ are inverses of each other. $\square$

1.4. Proof of $|\mathcal{SEFT}_n| \simeq Fred_n$

Fix an integer n and a Hilbert space $\mathcal{H}$ with a stable, orthogonal (see footnote 14) graded Cl_{-n}-action. Consider the categories $\mathcal{SEFT}_n = \mathcal{SEFT}_n(\mathcal{H})$, $\mathcal{V}_n = \mathcal{V}_n(\mathcal{H})$ and the space $Fred_n = Fred_n(\mathcal{H})$ defined in the previous sections.

Comparing $\mathcal{SEFT}_n$ and $\mathcal{V}_n$. We define a functor

$$\mathrm{ind}_n : \mathcal{SEFT}_n \to \mathcal{V}_n.$$

Let E be an object of $\mathcal{SEFT}_n$, and $V_\lambda^E = \ker(Q_E - \lambda)$ the eigenspaces of the generator Q_E of E (see section 1.2). On objects, ind_n is given by

$$\mathrm{ind}_n E = V_0^E.$$

For a morphism (α, f, A) from an object E to another object E' in $\mathcal{SEFT}_n$, we define

$$\mathrm{ind}_n(\alpha, f, A) = \Big(f|_{V_0^E}, f\big(\bigoplus_{\substack{\lambda > 0 \\ \alpha(\lambda) = 0}} V_\lambda^E \big) \oplus (A \cap V_0^{E'}) \Big).$$

The right hand side is indeed a morphism from $\mathrm{ind}_n E$ to $\mathrm{ind}_n E'$ in $\mathcal{V}_n$ because

$$\begin{aligned}
\mathrm{ind}_n E' &= V_0^{E'} = f\big(\bigoplus_{\alpha(\lambda)=0} V_\lambda^E \big) \oplus \big((A \oplus \epsilon A) \cap V_0^{E'} \big) \\
&= f(V_0^E) \oplus f\big(\bigoplus_{\substack{\lambda>0 \\ \alpha(\lambda)=0}} V_\lambda^E \big) \oplus f\big(\bigoplus_{\substack{\lambda<0 \\ \alpha(\lambda)=0}} V_\lambda^E \big) \oplus (A \cap V_0^{E'}) \oplus (\epsilon A \cap V_0^{E'}) \\
&= f(\mathrm{ind}_n E) \oplus f\big(\bigoplus_{\substack{\lambda>0 \\ \alpha(\lambda)=0}} V_\lambda^E \big) \oplus \epsilon f\big(\bigoplus_{\substack{\lambda>0 \\ \alpha(\lambda)=0}} V_\lambda^E \big) \oplus \big(A \cap V_0^{E'} \big) \oplus \epsilon \big(A \cap V_0^{E'} \big).
\end{aligned}$$

An example (with $A = 0$) is depicted in figure 3. It is straightforward to check that ind_n respects compositions.

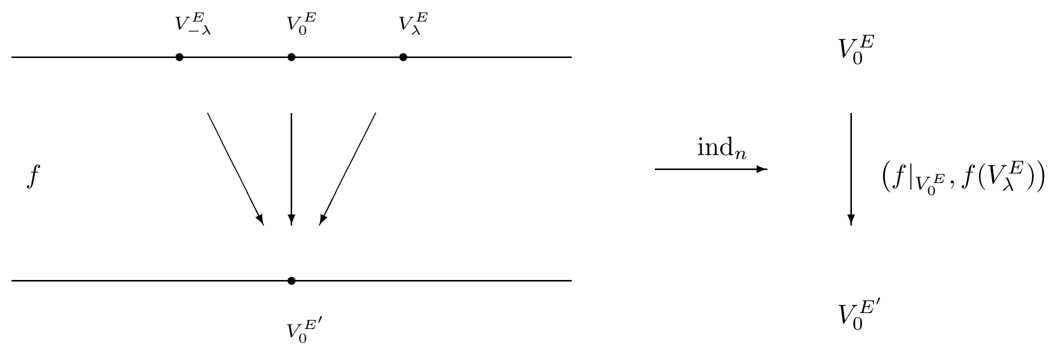

Figure 3: The functor ind_n on a particular morphism in $\mathcal{SEFT}_n$.

Proposition 1.4.1. *The functor ind_n induces a homotopy equivalence*

$$|ind_n| : |\mathcal{SEFT}_n| \xrightarrow{\sim} |\mathcal{V}_n|.$$

Proof. We first define a functor $\mathcal{E}_n : \mathcal{V}_n \hookrightarrow \mathcal{SEFT}_n$ which identifies $\mathcal{V}_n$ with a subcategory of $\mathcal{SEFT}_n$. Given a graded Cl_{-n}-submodule $V \subset \mathcal{H}$, the 1-SEFT $\mathcal{E}_n(V)$ is defined by $\mathcal{E}_n(V)(Z_{1,0}) = \mathcal{H}$ and

$$\mathcal{E}_n(V)_B(I_{z,\theta}) = \begin{cases} 1 & \text{on } V \\ 0 & \text{on } V^\perp \subset \mathcal{H} \end{cases},$$

for any object B of $\mathcal{S}$ and $(z, \theta) \in \mathcal{O}(B)^{\mathrm{even}}_{(0,\infty)} \times \mathcal{O}(B)^{\mathrm{odd}}$. If $V \neq 0$, $\operatorname{sp} Q_{\mathcal{E}_n(V)} = \{0\}$ and $V_0^{\mathcal{E}_n(V)} = V$; if $V = 0$, $\operatorname{sp} Q_{\mathcal{E}_n(V)}$ is empty. For a morphism (f, A) from an object V to another object V' in $\mathcal{V}_n$, we have

$$\mathcal{E}_n(f, A) = (\alpha_{V,V'}, f, A),$$

where $\alpha_{V,V'}$ is the inclusion of $\operatorname{sp} Q_{\mathcal{E}_n(V)}$ into $\operatorname{sp} Q_{\mathcal{E}_n(V')}$.

The composition $\mathrm{ind}_n \mathcal{E}_n$ is the identity on $\mathcal{V}_n$, as it is clear that

$$\begin{aligned} \mathrm{ind}_n \mathcal{E}_n(V) &= V_0^{\mathcal{E}_n(V)} = V, \\ \mathrm{ind}_n \mathcal{E}_n(f, A) &= \mathrm{ind}_n(\alpha_{V,V'}, f, A) = (f, A). \end{aligned}$$

On the other hand, there is a natural transformation N from $\mathcal{E}_n \mathrm{ind}_n$ to the identity on $\mathcal{SEFT}_n$,

$$N(E) = \Big(\alpha_E, i_E, \bigoplus_{\lambda > 0} V_\lambda^E\Big),$$

where α_E is the inclusion of $\{0\}$ or $\emptyset$ into $\operatorname{sp} Q_E$, depending on whether $0 \in \operatorname{sp} Q_E$ or not, and i_E is the inclusion of V_0^E (possibly 0) into $\bigoplus_\lambda V_\lambda^E$. It is clear the right hand side above is indeed a morphism from

$\mathcal{E}_n \mathrm{ind}_n(E) = \mathcal{E}_n(V_0^E)$ to E. It remains to check the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{E}_n \mathrm{ind}_n(E) \arrow[r, "N(E)"] \arrow[d, "\mathcal{E}_n \mathrm{ind}_n(\alpha,f,A)"'] & E \arrow[d, "(\alpha,f,A)"] \\
\mathcal{E}_n \mathrm{ind}_n(E') \arrow[r, "N(E')"] & E'
\end{tikzcd}$$

for each morphism (α, f, A) in $\mathcal{SEFT}_n$. On one hand, we have

$$(\alpha, f, A) \circ N(E) = \Big(\beta, f|_{V_0^E}, f(\bigoplus_{\lambda>0} V_\lambda^E) \oplus A\Big),$$

where β is the inclusion of $\{0\}$ or $\emptyset$ into $\mathrm{sp}\, Q_{E'}$, depending on whether $0 \in \mathrm{sp}\, Q_E$ or not. On the other hand,

$$N(E') \circ \mathcal{E}_n \mathrm{ind}_n(\alpha, f, A) = \Big(\beta, f|_{V_0^E}, f(\bigoplus_{\substack{\lambda>0 \\ \alpha(\lambda)=0}} V_\lambda^E) \oplus (A \cap V_0^{E'}) \oplus \bigoplus_{\lambda'>0} V_{\lambda'}^{E'}\Big).$$

Because $A = \bigoplus_{\lambda' \geq 0} A \cap V_{\lambda'}^E$ and $f(\bigoplus_{\alpha(\lambda)=\lambda'} V_\lambda^E) \oplus (A \cap V_{\lambda'}^{E'}) = V_{\lambda'}^{E'}$ for each $\lambda' > 0$, the two compositions are the same. We have shown that ind_n induces a deformation retraction of $|\mathcal{SEFT}_n|$ onto $|\mathcal{V}_n|$. $\square$

Comparing $|\mathcal{V}_n|$ and $Fred_n$. To finish the proof of theorem 1.2.3, it remains to establish the following

Proposition 1.4.2. *There is a homotopy equivalence*

$$|\mathcal{V}_n| \simeq Fred_n.$$

Remark. In [Seg77], Segal outlined the proofs of the homotopy equivalences

$$|\widehat{Vect}| \simeq Fred_0, \qquad |QVect| \simeq Fred_1.$$

According to propositions 1.3.2 and 1.3.3, these imply the cases $n = 0, 1$ of proposition 1.4.2. In essence, we are going to generalize Segal's arguments.

Proof. Define a topological category $\mathcal{C}$ as follows. An object (V, i) of $\mathcal{C}$ consists of an object V of $\mathcal{V}_n$ and an even, Cl_{-n}-linear, isometric embedding $i : V \hookrightarrow \mathcal{H}$. Let $\mathrm{Emb}'(V, \mathcal{H})$ denote the space of such embeddings. A morphism (V, V', f, A, i) in $\mathcal{C}$ consists of two objects V, V' of $\mathcal{V}_n$, a morphism (f, A) of $\mathcal{V}_n$ from V to V', and some $i \in \mathrm{Emb}'(V', \mathcal{H})$. Such a 5-tuple is a morphism from (V, if) to (V', i). A pair of composable morphisms must be of the form (V, V', f, A, if') and (V', V'', f', A', i), whose composition is $(V, V'', f'f, f'(A) \oplus A', i)$.

Lemma 1.4.3. $|\mathcal{V}_n| \simeq |\mathcal{C}|$.

Proof. There is a functor from $\mathcal{C}$ to $\mathcal{V}_n$ defined by $(V,i)\mapsto V$ and $(V,V',f,A,i)\mapsto(f,A)$. One easily sees that, for any $p\in\mathbb{Z}_+$, the induced map from $N_p\mathcal{C}$ to $N_p\mathcal{V}_n$ is a fiber bundle whose fibers are $\mathrm{Emb}'(V,\mathcal{H})$ for various V. [17] Since $\mathrm{Emb}'(V,\mathcal{H})\simeq *$, the functor induces a homotopy equivalence of the classifying spaces. □

Now, we define another topological category $\mathcal{C}'$. The objects of $\mathcal{C}'$ are the same as those of $\mathcal{V}_n$, i.e. finite dimensional, graded Cl_{-n}-submodules of $\mathcal{H}$. However, $N_0\mathcal{C}'$ is topologized as a subspace of $\mathrm{Gr}(\mathcal{H})$. A morphism of $\mathcal{C}'$ is a pair $(V,A)\in N_0\mathcal{C}'\times\mathrm{Gr}(\mathcal{H})$, where A is a finite dimensional Cl_{-n}-submodule such that $A\perp V$ and $A\perp\epsilon A$; it is a morphism from V to $V\oplus A\oplus\epsilon A$. Two composable morphisms must be of the form (V,A) and $(V\oplus A\oplus\epsilon A,A')$, whose composition is $(V,A\oplus A')$.

Lemma 1.4.4. $|\mathcal{C}|\simeq|\mathcal{C}'|$.

Proof. Let $\mathcal{C}_{\mathrm{iso}}\subset\mathcal{C}$ be the subcategory with the same objects but only those morphisms of the form $(V,V',f,0,i)$. A point x in $|\mathcal{C}_{\mathrm{iso}}|$ is uniquely represented by the following data,

$$\left(V_0\xrightarrow{f_1}V_1\xrightarrow{f_2}\cdots\xrightarrow{f_p}V_p\xrightarrow{i}\mathcal{H}\ ,\ \vec{s}\in\Delta^p-\partial\Delta^p\right), \tag{1.4.5}$$

where f_i are isomorphisms. We use the following notations for some of the data associated to x: $p^x=p$, $V_i^x=V_i$, $f^x_{\mathrm{comp}}=f_p\cdots f_2f_1$, $i^x=i$, and $\vec{s}^{\,x}=\vec{s}$.

We define a category $\mathcal{D}$ such that $N_0\mathcal{D}=|\mathcal{C}_{\mathrm{iso}}|$ and $|\mathcal{D}|\cong|\mathcal{C}|$. For $x,y\in|\mathcal{C}_{\mathrm{iso}}|$, a morphism in $\mathcal{D}$ from x to y is a 4-tuple (x,f,A,y), where (f,A) is a morphism in $\mathcal{V}_n$ from $V^x_{p^x}$ to V^y_0, such that $i^x=i^y\circ f^y_{\mathrm{comp}}\circ f$. Composition is defined as follows,

$$(y,f',A',z)\circ(x,f,A,y)=\big(x,\ f'\circ f^y_{\mathrm{comp}}\circ f,\ f'\circ f^y_{\mathrm{comp}}(A)\oplus A',\ z\big).$$

We construct a map $\phi:|\mathcal{D}|\to|\mathcal{C}|$. The restriction of ϕ on $N_0\mathcal{D}=|\mathcal{C}_{\mathrm{iso}}|$ is the map induced by the inclusion $\mathcal{C}_{\mathrm{iso}}\subset\mathcal{C}$. For $q\geq1$, a point in $N_q\mathcal{D}\times\Delta^q$ has the form $\big(x_0,f_1,A_1,x_1,f_2,A_2,\ldots,f_q,A_q,x_q;\vec{t}\,\big)$, where $\vec{t}\in\Delta^q$ and the i-th morphism of the q-chain is (x_{i-1},f_i,A_i,x_i). Its image under ϕ is defined to be

$$\left(V^{x_0}_\bullet\xrightarrow{(f_1,A_1)}V^{x_1}_\bullet\xrightarrow{(f_2,A_2)}\cdots\xrightarrow{(f_q,A_q)}V^{x_q}_\bullet\xrightarrow{i^{x_q}}\mathcal{H}\ ,\ (t_0\vec{s}^{\,x_0},\ldots,t_q\vec{s}^{\,x_q})\right)\in|\mathcal{C}|.$$

It is straightforward to check that ϕ is well-defined, i.e. compatible with faces and degeneracies, and is a homemorphism.

There is a functor $G:\mathcal{D}\to\mathcal{C}'$, defined by

$$N_0G(x)=i^x(V^x_{p^x}),\qquad N_1G(x,f,A,y)=\big(i^x(V^x_{p^x}),i^y\circ f^y_{\mathrm{comp}}(A)\big).$$

First, we show that N_0G is a fiber bundle with contractible fibers. [18] Fix $v\in N_0\mathcal{C}'$. Let $U^{\mathrm{even}}_{Cl_{-n}}(\mathcal{H})$ be the space of even, Cl_{-n}-linear, unitary operators on $\mathcal{H}$. Since the map from $U^{\mathrm{even}}_{Cl_{-n}}(\mathcal{H})$ to $N_0\mathcal{C}'$ given by $T\mapsto T(v)$ is a fiber bundle, it admits a section $V\mapsto T_V$ for V in a neighhorhood $U\subset N_0\mathcal{C}'$ of v. Notice that $T_V(v)=V$. There is a homeomorphism $(N_0G)^{-1}(U)\cong U\times(N_0G)^{-1}(v)$; for any $V\in U$, the image of $x\in(N_0G)^{-1}(V)$ is (V,x'), where the data comprising $x'\in(N_0G)^{-1}(v)$ – see (1.4.5) – are the same as those of x except with $i^{x'}=T_V^{-1}\circ i^x$. The fiber $(N_0G)^{-1}(v)$ is the classifying space of $\mathcal{C}_v\subset\mathcal{C}_{\mathrm{iso}}$, which is

[17] As usual, $N_\bullet$ denotes the nerve functor on small categories.

[18] In fact, $|\mathcal{C}_{\mathrm{iso}}|\simeq\coprod_{[V]}BU^{\mathrm{even}}_{Cl_{-n}}(V)\simeq N_0\mathcal{C}'$, where V is any finite dimensional, graded Cl_{-n}-module, and the disjoint union has one component for each isomorphism class of such modules.

the full subcategory consisting of objects (V,i) such that $i(V)=v$. Any object of $\mathcal{C}_v$ is both initial and terminal, hence $(N_0G)^{-1}(v)=|\mathcal{C}_v|\simeq *$.

Similarly, N_1G is also a fiber bundle with contractible fibers. Fix $(v,a)\in N_1\mathcal{C}'$. Let U be a neighborhood of (v,a) that admits a continuous map to $U^{\text{even}}_{Cl_{-n}}(\mathcal{H})$, given by $(V,A)\mapsto T_{V,A}$, such that $T_{V,A}(v)=V$, $T_{V,A}(a)=A$, $\forall\,(V,A)\in U$. There is a homeomorphism $(N_1G)^{-1}(U)\cong U\times(N_1G)^{-1}(v,a)$ defined as follows: $\forall\,(V,A)\in U$, $(x,f,A,y)\in(N_1G)^{-1}(V,A)$ is mapped to $(x',f,A,y')\in(N_1G)^{-1}(v,a)$, where the data comprising x' (resp. y') are the same as those of x (resp. y) except with $i^{x'}=T^{-1}_{V,A}\circ i^x$ (resp. $i^{y'}=T^{-1}_{V,A}\circ i^y$). Now, consider the fiber $(N_1G)^{-1}(v,a)$. Notice that, for any $(x,f,A,y)\in(N_1G)^{-1}(v,a)$, $x\in(N_0G)^{-1}(v)$ and $y\in(N_0G)^{-1}(v')$, where $v'=v\oplus a\oplus\epsilon a$. Let $x_0=(v,\text{inc})\in N_0\mathcal{C}_v$ and $y_0=(v',\text{inc})\in N_0\mathcal{C}_{v'}$. Since all objects of $\mathcal{C}_v$ and $\mathcal{C}_{v'}$ are initial, there are canonical contractions of $(N_0G)^{-1}(v)=|\mathcal{C}_v|$ and $(N_0G)^{-1}(v')=|\mathcal{C}_{v'}|$ to the points x_0 and y_0 respectively. These two contractions uniquely determine a contraction of $(N_1G)^{-1}(v,a)$ to the point (x_0,inc,a,y_0), so that $(N_1G)^{-1}(v,a)\simeq *$.

We have shown that N_0G and N_1G are homotopy equivalences. Since the face maps $N_1\mathcal{D}\rightrightarrows N_0\mathcal{D}$, $N_1\mathcal{C}'\rightrightarrows N_0\mathcal{C}'$ are fibrations, $|G|:|\mathcal{D}|\to|\mathcal{C}'|$ is also a homotopy equivalence. □

Let us define yet another topological category $\mathcal{C}''$ as follows:

$$\begin{aligned} N_0\mathcal{C}'' &= \big\{(V,F)\in N_0\mathcal{C}'\times Fred_n|F(V)\subset V,\ker F\subset V\big\},\\ N_1\mathcal{C}'' &= \big\{(V,A,F)\in N_1\mathcal{C}'\times Fred_n|F(V)\subset V,F(A)\subset A,\ker F\subset V,F|_A>0\big\},\end{aligned}$$

where $(V,A,F)\in N_1\mathcal{C}''$ is a morphism from (V,F) to $(V\oplus A\oplus\epsilon A,F)$. Hence, two morphisms are composable iff they are of the form (V,A,F) and $(V\oplus A\oplus\epsilon A,A',F)$, and their composition is defined to be $(V,A\oplus A',F)$.

Lemma 1.4.6. $|\mathcal{C}'|\simeq|\mathcal{C}''|$.

Proof. Define a functor $\pi_1:\mathcal{C}''\to\mathcal{C}'$ by $(V,F)\mapsto V$, $(V,A,F)\mapsto(V,A)$. It suffices to show that $N_p\pi_1$ is a homotopy equivalence for every $p\geq 0$. A p-chain of $\mathcal{C}''$ whose i-th morphism is (V_{i-1},A_i,F), $i=1,\dots,p$, is determined by the $(p+2)$-tuple $(V_0,A_1,\dots,A_p,F)$; we represent any p-chain of $\mathcal{C}''$ by such a $(p+2)$-tuple. Analogously, we represent a p-chain of $\mathcal{C}'$ by a $(p+1)$-tuple $(V,A_1,\dots,A_p)$.

Fix $(v,a_1,\dots,a_p)\in N_p\mathcal{C}'$. Since $T\mapsto(T(v),T(a_1),\dots,T(a_p))$ defines a fiber bundle $U^{\text{even}}_{Cl_{-n}}(\mathcal{H})\to N_p\mathcal{C}'$, $(v,a_1,\dots,a_p)$ has a neighhorhood U in $N_p\mathcal{C}'$ such that, $\forall\,C=(V,A_1,\dots,A_p)\in U$, $\exists\,T_C\in U^{\text{even}}_{Cl_{-n}}(\mathcal{H})$ which depends continuously on C and satisfies $T_C(v)=V$, $T_C(a_i)=A_i$, $i=1,\dots,p$. Denote the vector space $v\oplus a_1\oplus\epsilon a_1\oplus\cdots\oplus a_p\oplus\epsilon a_p$ by v_p. Observe there is a homeomorphism

$$\begin{array}{ccc} (N_p\pi_1)^{-1}(U) & \cong & U\times\operatorname{End}'(v)\times\prod_{i=1}^p\operatorname{End}'_+(a_i)\times GL'(v_p^\perp)\\ (C,F) & \mapsto & (C,\ T_C^{-1}FT_C|_v,\ T_C^{-1}FT_C|_{a_1},\dots,T_C^{-1}FT_C|_{a_p},\ T_C^{-1}FT_C|_{v_p^\perp}).\end{array}\tag{1.4.7}$$

Here, $\operatorname{End}'(v)$ is the vector space of odd, self-adjoint, Cl_{-n}-linear endomorphisms of v; $\operatorname{End}'_+(a_i)$ is the space of positive definite, Cl_{-n}-linear endomorphisms of a_i; $GL'(v_p^\perp)$ is the space of invertible operators on $v_p^\perp$ which are odd, self-adjoint, Cl_{-n}-linear and, if $n\equiv 1\pmod 4$, condition (1.2.2) is satisfied. Notice that $\mathcal{H}=v\oplus a_1\oplus\epsilon a_1\oplus\cdots\oplus a_p\oplus\epsilon a_p\oplus v_p^\perp$, hence the odd operator $T_C^{-1}FT_C$, or F, is determined by its restrictions on v, $a_1,\dots,a_p$ and $v_p^\perp$. Because of the homeomorphism (1.4.7), $N_p\pi_1$ is a fiber bundle whose fiber is the product of $\operatorname{End}'(v)$, $\operatorname{End}'_+(a_i)$, $i=1,\dots,p$, and $GL'(v_p^\perp)$.

It remains to show that the fibers of $N_p\pi_1$ are contractible. Each $\operatorname{End}'_+(a_i)$ is a convex subspace of a vector space, thus contractible. Let us consider $GL'(\mathcal{H}')$, where $\mathcal{H}'$ is any Hilbert space with a stable graded Cl_{-n}-action, e.g. $v_p^\perp$. In the case $n=0$, $T\mapsto T|_{\mathcal{H}'^{\text{even}}}$ defines a homeomorphism $GL'(\mathcal{H}')\cong$

$GL(\mathcal{H}'^{\text{even}}, \mathcal{H}'^{\text{odd}})$, and the latter space is contractible by Kuiper's theorem [Kui65]. In the case $n < 0$, let $e_1, \ldots, e_{|n|}$ be an orthonormal basis of $\mathbb{R}^{|n|}$, regarded as elements of Cl_{-n}. There is an action of $Cl_{-n}^{\text{even}} \cong Cl_{-n-1}$ on $\mathcal{H}'^{\text{even}}$. The map $T \mapsto (e_1 T)|_{\mathcal{H}'^{\text{even}}}$ defines a homeomorphism from $GL'(\mathcal{H}')$ to $GL''(\mathcal{H}'^{\text{even}})$, the space of skew-adjoint, Cl_{-n-1}-antilinear, [19] invertible operators S on $\mathcal{H}'^{\text{even}}$ with the property that, if $n \equiv 1 \pmod 4$, $e_1 e_2 \cdots e_{|n|-1} S$ is not essentially positive or negative. It follows from Kuiper's theorem again that $GL''(\mathcal{H}'^{\text{even}}) \simeq *$. [AS69] Finally, if $n > 0$, suppose $n = 4k - \ell$ where $k > 0$ and $\ell \geq 0$ are integers. Let M be an irreducible graded $Cl_{\ell,\ell}$-module. Since $Cl_{0,n} \otimes Cl_{\ell,\ell} \cong Cl_{\ell,4k} \cong Cl_{4k+\ell}$ as a graded algebra, $\mathcal{H}' \otimes M$ admits a stable graded $Cl_{4k+\ell}$-action. The map in (1.2.4) defines a homeomorphism $GL'(\mathcal{H}') \cong GL'(\mathcal{H}' \otimes M)$, where the latter space has just been shown to be contractible. □

Lemma 1.4.8. $|\mathcal{C}''| \simeq Fred_n$.

Proof. Regarding $Fred_n$ as a topological category with only identity morphisms, there is a functor $\pi_2 : \mathcal{C}'' \to Fred_n$ defined by $(V, F) \mapsto F$, $(V, A, F) \mapsto \mathrm{id}_F$. Our first step is to construct continuous local sections of $N_0 \pi_2 : N_0 \mathcal{C}'' \to Fred_n$. This will also give rise to local sections of $|\pi_2|$, via the inclusion $N_0 \mathcal{C}'' \to |\mathcal{C}''|$.

Fix $F_0 \in Fred_n$. Let $4c = ||(F_0|_{\ker F_0^\perp})^{-1}||^{-1} = \inf\big(\operatorname{sp} F_0 \cap (0, \infty)\big)$, which is a positive number, [20] and $U = \{F \in Fred_n : ||F - F_0|| < c\}$. For each $F \in U$, let e_F be the resolution of $1_{\mathcal{H}}$ associated with F, and define

$$p_F = e_F\big((-c, c)\big) \in B(\mathcal{H}), \qquad V_F = p_F(\mathcal{H}). \tag{1.4.9}$$

Notice that

$$1 - p_F = e_F\big((-\infty, -3c) \cup (3c, \infty)\big), \quad V_F^\perp = (1 - p_F)(\mathcal{H}). \tag{1.4.10}$$

(See [Rui91], §10.20.) We claim $F \mapsto (V_F, F)$ defines a continuous section of $N_0 \pi_2$ over U. This means, for $F \in U$,

(A) $\dim V_F < \infty$,

(B) $V_F \subset \mathcal{H}$ is a graded Cl_{-n}-submodule,

(C) $F(V_F) \subset V_F$,

(D) $\ker F \subset V_F$,

(E) the map $F \mapsto p_F$ is continuous.

Among these, (B), (C) and (D) follow immediately from (1.4.9).

Consider (A). It is true for $F = F_0$, because $V_{F_0} = \ker F_0$. To prove it for all $F \in U$, it suffices to show that, $\forall\, F_1, F_2 \in U$, p_{F_2} is injective on V_{F_1}. Let us assume the contrary, i.e. for some $F_1, F_2 \in U$, $\exists\, u \in V_{F_1}$ such that $||u|| = 1$ and $p_{F_2}(u) = 0$. Since $p_{F_1}(u) = u = (1 - p_{F_2})(u)$, by (1.4.9) and (1.4.10), we have $||F_1(u)|| < c$ and $||F_2(u)|| > 3c$, hence $||(F_2 - F_1)(u)|| > 2c$. However, $||F_i - F_0|| < c$, $i = 1, 2$, imply $||F_2 - F_1|| < 2c$, yielding a contradiction.

To prove (E), we fix $F_1 \in U$, $0 < \epsilon < 1$, and try to show $||p_F - p_{F_1}|| \leq \epsilon$ for any $F \in U$ with $||F - F_1|| < c\epsilon$. It suffices to obtain $||p_F(u) - p_{F_1}(u)|| \leq \epsilon$ for the following two cases:

(i) $u \in V_{F_1}$ is an eigenvector of F_1, $||u|| = 1$,

(ii) $u \perp V_{F_1}$, $||u|| = 1$.

[19] An operator T on a Cl_k-module V is Cl_k-antilinear if $Tv = -vT$ for any $v \in \mathbb{R}^{|k|} \subset Cl_k$.

[20] Since $\operatorname{im} F_0 = \ker F_0^\perp$, the restriction of F_0 to $\ker F_0^\perp$ has a bounded inverse by the open mapping theorem.

For case (i), suppose $F_1 u = \lambda u$ and $u = u' + u''$, where $u' = p_F(u) \in V_F$, $u'' \perp V_F$. Notice that $|\lambda| < c$ and $p_{F_1}(u) - p_F(u) = u - u' = u''$. The desired inequality $||u''|| \leq \epsilon$ follows from:

$$\begin{array}{rclcl} c\epsilon & > & ||F(u) - F_1(u)|| & = & ||F(u') + F(u'') - \lambda u' - \lambda u''|| \\ & \geq & ||F(u'') - \lambda u''|| & \geq & ||F(u'')|| - |\lambda|\,||u''|| \\ & > & 3c||u''|| - c||u''|| & = & 2c||u''||, \end{array}$$

where the fourth inequality follows from (1.4.10), and the second inequality from the fact that $F(u') - \lambda u' \in V_F$ and $F(u'') - \lambda u'' \in V_F^{\perp}$ are orthogonal. For case (ii), again write u as $u' + u''$, where $u' = p_F(u)$. Notice that $p_F(u) - p_{F_1}(u) = u' - 0 = u' \in V_F$. Let v' be an arbitrary vector in V_F with $||v'|| = 1$. By the proof of (A), $v' = p_F(v)$ for a unique $v \in V_{F_1}$. Write v as $v' + v''$; notice that $v'' \perp V_F$. From the proof of (i), we have $||v - v'|| < \epsilon ||v||/2 < ||v||/2$, implying $||v|| < 2$. Because $u \perp v$, we have $\langle u', v' \rangle + \langle u'', v'' \rangle = 0$, hence

$$|\langle u', v' \rangle| \leq ||u''||\,||v''|| \leq ||p_{F_1}(v) - p_F(v)|| \leq \frac{\epsilon}{2}\,||v|| < \epsilon,$$

where the third inequality follows from case (i). Since $||u'||$ equals the supremum of $|\langle u', v' \rangle|$ over all $v' \in V_{F_1}$ with $||v'|| = 1$, we have $||p_F(u) - p_{F_1}(u)|| = ||u'|| \leq \epsilon$, as desired. This finishes the proof of our claim that $F \mapsto (V_F, F)$ is a continuous section of $N_0\pi_2$, or $|\pi_2|$, over U.

Next, keeping the same notations, we show that $|\pi_2|^{-1}(U)$ can be contracted to the subspace $s_U = \{(V_F, F) | F \in U\}$ along the fibers of $|\pi_2|$. Since open sets U with such contractions form a base of $Fred_n$, this will prove $|\pi_2|$ is a homotopy equivalence by [DT58].

Let $\mathcal{C}''_{U,s_U} \subset \mathcal{C}''_U$ be the full subcategories of $\mathcal{C}''$ such that, given $(V, F) \in N_0\mathcal{C}''$,

$$\begin{array}{lcl} (V, F) \in N_0\mathcal{C}''_U & \Leftrightarrow & F \in U, \\ (V, F) \in N_0\mathcal{C}''_{U,s_U} & \Leftrightarrow & F \in U \text{ and } V \supset V_F. \end{array}$$

There is a functor $r : \mathcal{C}''_U \to \mathcal{C}''_{U,s_U}$, defined by

$$r(V, F) = (V + V_F, F), \qquad r(V, A, F) = (V + V_F, A \cap V_F^{\perp}, F).$$

Suppose $(V, F) \in N_0\mathcal{C}''_U$ and p is the orthogonal projection onto V. Since V, $V^{\perp}$ and V_F are invariant subspaces of F, the orthogonal projection onto $V + V_F = V \oplus (V_F \cap V^{\perp})$ is given by $p + p_F(1 - p) = p + p_F - pp_F$. Clearly, $(p, F) \mapsto p + p_F - pp_F$ is continuous, thus so is $(V, F) \mapsto V + V_F$. A similar argument applies to $(A, F) \mapsto A \cap V_F^{\perp}$. This shows r is continuous. The composition $r \circ \mathrm{inc}$ is the identity on $\mathcal{C}''_{U,s_U}$, where inc is the inclusion of $\mathcal{C}''_{U,s_U}$ into $\mathcal{C}''_U$. On the other hand, for any $(V, F) \in N_0\mathcal{C}''$, let $A_{V,F} = e_F\big((0, \infty)\big)(V^{\perp})$, which depends continuously on (V, F). There is a natural transformation from the identity on $\mathcal{C}''_U$ to $\mathrm{inc} \circ r$, given by $(V, F) \mapsto (V, V_F \cap A_{V,F}, F)$. Indeed, for any morphism (V, A, F) of $\mathcal{C}''_U$, the following diagram commutes:

$$\begin{tikzcd} (V, F) \arrow[r, "{(V, V_F \cap A_{V,F}, F)}"] \arrow[d, "{(V, A, F)}"'] & (V + V_F, F) \arrow[d, "{(V + V_F, A \cap V_F^{\perp}, F)}"] \\ (V', F) \arrow[r, "{(V', V_F \cap A_{V',F}, F)}"] & (V' + V_F, F), \end{tikzcd}$$

where $V' = V \oplus A \oplus \epsilon A$. Commutativity follows from the fact that there is at most one morphism between any two objects of $\mathcal{C}''$. Therefore, $|r|$ defines a deformation retraction of $|\pi_2|^{-1}(U) = |\mathcal{C}''_U|$ onto the subspace $|\mathcal{C}''_{U,s_U}|$.

Finally, any object (V, F) of $\mathcal{C}''_{U,s_U}$ admits a unique morphism $(V_F, V \cap A_{V_F,F}, F)$ from (V_F, F), which depends continuously on (V, F). In other words, each $(V_F, F) \in s_U$ is an initial object of a preimage category $\pi_2^{-1}(F)$, and the associated contraction of $|\pi_2^{-1}(F)| = |\pi_2|^{-1}(F)$ to the point (V_F, F) depends continuously on $F \in U$. Therefore, $|\mathcal{C}''_{U,s_U}|$ deformation retracts onto s_U. Notice that the fibers of $|\pi_2|$ are preserved during both of the deformations constructed above. $\square$

This completes the proof of proposition 1.4.2. $\square$

CHAPTER 2: ANNULAR FIELD THEORIES AND THE ELLIPTIC COHOMOLOGY FOR THE TATE CURVE

2.1. Annular Field Theories

Some super manifolds of dimensions $(1|1)$ and $(2|1)$. Consider the $(1|1)$-manifold

$$S = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{0|1}.$$

Let (s, λ) be the standard coordinates on S, where s is defined only locally. The metric $ds + \lambda d\lambda$ on $\mathbb{R}^{1|1}$ defined right above (1.1.3) descends to a metric

$$\omega^S = ds + \lambda d\lambda$$

on $S^{1|1}$. Also consider the $(2|1)$-manifolds

$$A = \mathbb{R}/\mathbb{Z} \times \mathbb{R} \times \mathbb{R}^{0|1}, \qquad A_t = \mathbb{R}/\mathbb{Z} \times [0, t] \times \mathbb{R}^{0|1},$$

with standard coordinates (s, s', λ). Let

$$\omega_1^A = ds + ids', \qquad \omega_2^A = ds - ids' + \lambda d\lambda,$$

which are complex 1-forms on A or A_t. Define the following maps between these super manifolds:

$$\begin{array}{lll} & \tau_t : S \to S, & \tau_t^*(s, \lambda) = (s + t, \lambda), \\ & \nu_{t,t'} : S \to A, & \nu_{t,t'}^*(s, s', \lambda) = (s + t, t', \lambda), \\ & \kappa_{t,t'} : A \to A, & \nu_{t,t'}^*(s, s', \lambda) = (s + t, s' + t', \lambda), \\ & \epsilon : S \to S, & \epsilon^*(s, \lambda) = (s, -\lambda), \\ \text{or} & \epsilon : A \to A, & \epsilon^*(s, s', \lambda) = (s, s', -\lambda), \end{array}$$

where $t \in \mathbb{R}/\mathbb{Z}$, $t' \in \mathbb{R}$. Simple calculations then show the following:

$$\operatorname{Hom}(S, \omega^S; S, \omega^S) = \{\tau_t, \tau_t\epsilon\}, \tag{2.1.1}$$

$$\operatorname{Hom}(S, ds, \omega^S; A, \omega_1^A, \omega_2^A) = \{\nu_{t,t'}, \nu_{t,t'}\epsilon\}, \tag{2.1.2}$$

$$\operatorname{Hom}(A, \omega_1^A, \omega_2^A; A, \omega_1^A, \omega_2^A) = \{\kappa_{t,t'}, \kappa_{t,t'}\epsilon\}. \tag{2.1.3}$$

The notation in e.g. (2.1.2) refers the set of maps from S to A that pull back ω_1^A and ω_2^A to $ds = \omega^S|_{S_{\mathrm{red}}}$ and ω^S respectively.

Let B be a super manifold with $B_{\mathrm{red}} =$ a point, and $\Lambda = \mathcal{O}(B)$. The natural transformations (s_B, λ_B) and (s_B, s'_B, λ_B) — see (1.1.8) — allow us to make the identifications

$$S(B) = \Lambda^{\mathrm{even}}/\mathbb{Z} \times \Lambda^{\mathrm{odd}}, \quad A(B) = \Lambda^{\mathrm{even}}/\mathbb{Z} \times \Lambda^{\mathrm{even}} \times \Lambda^{\mathrm{odd}},$$

where $\Lambda^{\mathrm{even}}/\mathbb{Z}$ is a quotient of abelian groups. Defining

$$\begin{array}{lll} & \tau_x : S(B) \to S(B), & (w, \eta) \mapsto (w + x, \eta), \\ & \nu_{x,y,\theta} : S(B) \to A(B), & (w, \eta) \mapsto (w + x - \theta\eta/2, y - i\theta\eta/2, \theta + \eta), \\ & \kappa_{x,y,\theta} : A(B) \to A(B), & (w, w', \eta) \mapsto (w + x - \theta\eta/2, w' + y - i\theta\eta/2, \theta + \eta), \\ & \epsilon : S(B) \to S(B), & (w, \eta) \mapsto (w, -\eta), \\ \text{or} & \epsilon : A(B) \to A(B), & (w, w', \eta) \mapsto (w, w', -\eta), \end{array}$$

[21] with $x \in \Lambda^{\text{even}}/\mathbb{Z}$, $y \in \Lambda^{\text{even}}$, $\theta \in \Lambda^{\text{odd}}$, we have

$$\operatorname{Hom}\big(S(B), ds_B, \omega_B^S; S(B), ds_B, \omega_B^S\big) = \{\tau_x, \tau_x\epsilon\}, \tag{2.1.4}$$
$$\operatorname{Hom}\big(S(B), ds_B, \omega_B^S; A(B), (\omega_1^A)_B, (\omega_2^A)_B\big) = \{\nu_{x,y,\theta}, \nu_{x,y,\theta}\epsilon\}, \tag{2.1.5}$$
$$\operatorname{Hom}\big(A(B), (\omega_1^A)_B, (\omega_2^A)_B, A(B), (\omega_1^A)_B, (\omega_2^A)_B\big) = \{\kappa_{x,y,\theta}, \kappa_{x,y,\theta}\epsilon\}. \tag{2.1.6}$$

Recalling (1.1.9), the notation in e.g. (2.1.5) refers to the set of maps $\nu : S(B) \to A(B)$ such that $(\omega_1^A)_B \circ T\nu = ds_B$ and $(\omega_2^A)_B \circ T\nu = \omega_B^S$. The proofs of (2.1.4)-(2.1.6) are similar to those of (1.1.11)-(1.1.13).

There are analogues of lemmas 1.1.15 and 1.1.27 for the maps in (2.1.4)-(2.1.6). More precisely, if $\phi : M_1(B) \to M_2(B)$ is any of those maps, there is a map $\phi_{\text{red}} : (M_1)_{\text{red}} \to (M_2)_{\text{red}}$ of ordinary manifolds such that $\phi(f)_{\text{red}} = \phi_{\text{red}} f_{\text{red}}$, $\forall f \in M_1(B)$, and also a map $\ell(\phi) : M_1 \to M_2$ of super manifolds such that $\ell(\phi)_{\text{red}} = \phi_{\text{red}}$. The latter ones are defined as follows:

$$\begin{aligned} &\ell(\tau_x) = \tau_{x^0}, && \ell(\tau_x\epsilon) = \tau_{x^0}\epsilon, \\ &\ell(\nu_{x,y,\theta}) = \nu_{x^0,y^0}, && \ell(\nu_{x,y,\theta}\epsilon) = \nu_{x^0,y^0}\epsilon, \\ &\ell(\kappa_{x,y,\theta}) = \kappa_{x^0,y^0}, && \ell(\kappa_{x,y,\theta}\epsilon) = \kappa_{x^0,y^0}\epsilon, \end{aligned} \tag{2.1.7}$$

where e.g x^0 is the unique real number so that $x - x^0$ is nilpotent. The induced maps ϕ_{red} and $\ell(\phi)$ respect compositions, i.e.

$$(\phi'\phi)_{\text{red}} = \phi'_{\text{red}}\phi_r, \quad \ell(\phi'\phi) = \ell(\phi')\ell(\phi),$$

whenever ϕ and ϕ' are composable.

Define $C(S)$ to be the Clifford algebra on the space of constant odd functions on S, i.e. $\mathbb{R}\lambda \subset \mathcal{O}(S)^{\text{odd}}$, with respect to the quadratic form $c\lambda \mapsto c^2$. Clearly, $C(S) \cong Cl_1$. Also, for each $t > 0$, let $F(A_t)$ be the exterior algebra on the space of constant odd functions on A_t. [22] $C(S)$ and $F(A_t)$ have the similar properties as the Clifford algebras and Fock spaces in section 1.1. Let us repeat them below in terms of our (1|1)- and (2|1)-manifolds. For details, refer to [ST04].

Consider a *bordism*

$$\begin{tikzcd} & (A_t, \omega_1^A, \omega_2^A) & \\ (S, ds, \omega^S) \arrow[ur, "\alpha"] & & (S, ds, \omega^S), \arrow[ul, "\beta"'] \end{tikzcd}$$

i.e. α_{red} (resp. β_{red}) maps S_{red} onto the $s' = 0$ (resp. $s' = t$) boundary of A_t; the notations in the diagram mean that α and β belong to (2.1.2). Any such bordism canonically induces a $C(S)$-$C(S)$-action on $F(A_t)$. In fact, $F(A_t)$ is irreducible and is generated by a special element Ω. It also has the property

[21] More precisely, $\nu_{x,y,\theta}$ and $\kappa_{x,y,\theta}$ are maps into $A(B) \otimes \mathbb{C}$.

[22] $C(S)$ and $F(A_t)$ are small parts of the corresponding Clifford algebra and Fock space defined in [ST04].

that $\lambda\Omega = \Omega\lambda$, where $\lambda \in C(S)$. If there is a commutative diagram

$$\begin{tikzcd}
& (A_t, \omega_1^A, \omega_2^A) \arrow[dd, "\kappa"'] & \\
(S, ds, \omega^S) \arrow[ur, "\alpha"] \arrow[dr, "\alpha'"'] & & (S, ds, \omega^S) \arrow[ul, "\beta"'] \arrow[dl, "\beta'"] \\
& (A_t, \omega_1^A, \omega_2^A) &
\end{tikzcd}$$

where the map κ is invertible, then each of the two bordisms in the diagram gives rise to a $C(S)$-$C(S)$-action on $F(A_t)$, and κ induces an isomorphism

$$\kappa_* : F(A_t) \xrightarrow{\sim} F(A_t)$$

of the two $C(S)$-$C(S)$-bimodules. Notice that by (2.1.3), $\kappa = 1$ or ϵ. Finally, suppose there is a diagram (with the 1-forms suppressed)

$$\begin{tikzcd}
& & A_{t_1+t_2} & & \\
& A_{t_1} \arrow[ur, "\iota"] & & A_{t_2} \arrow[ul, "\iota'"'] & \\
S \arrow[ur, "\alpha"] & & S \arrow[ul, "\beta"'] \arrow[ur, "\alpha'"] & & S, \arrow[ul, "\beta'"']
\end{tikzcd}$$

where the four maps on the bottom define two bordisms and the square is a pushout. The maps $\iota\alpha$ and $\iota'\beta'$ then also define a bordism, and there is a canonical isomorphism of $C(S)$-$C(S)$-bimodules,

$$F(A_{t_2}) \otimes_{C(S)} F(A_{t_1}) \cong F(A_{t_1+t_2}), \tag{2.1.8}$$

which sends $\Omega \otimes \Omega$ to Ω.

Field theories defined using S and A_t. We now construct a super category $\mathcal{SAB}_n$ for each $n \in \mathbb{Z}$. It is a 'super' version of a $(1+1)$-dimensional bordism category whose only bordisms are annuli. This should be thought of as a subcategory of a super category that incorporates all compact surfaces with conformal structures.

Fix a super manifold B with B_{red} = a point, and let $\Lambda = \mathcal{O}(B)$. The only object of $\mathcal{SAB}_n(B)$ is the $(1|1)$-manifold S. Like $\mathcal{SEB}_n^1(B)$, the morphisms are divided into 'decorated B-isomorphisms' and equivalence classes of 'decorated B-bordisms.' A *decorated B-isomorphism* (τ, c) consists of an invertible map

$$\tau : \big(S(B), \omega_B^S\big) \to \big(S(B), \omega_B^S\big),$$

together with an element $c \in C(S)^{\otimes -n}$. (See section 1.1, footnote 12.) A *decorated B-bordism* is a 4-tuple

$(A_t, \alpha, \beta, \Psi)$ consisting of a $(2|1)$-manifold A_t, maps

$$
\begin{tikzcd}
& \big(A_t(B), (\omega_1^A)_B, (\omega_2^A)_B\big) & \\
\big(S(B), ds_B, \omega_B^S\big) \arrow[ur, "\alpha"] & & \big(S(B), ds_B, \omega_B^S\big), \arrow[ul, "\beta"']
\end{tikzcd}
$$

and an element $\Psi \in F(A_t)^{\otimes -n}$. This 4-tuple is *equivalent* to those of the form $(A_t, \kappa\alpha, \kappa\beta, \ell(\kappa)_*\Psi)$, where

$$\kappa : \big(A_t(B), (\omega_1^A)_B, (\omega_2^A)_B\big) \to \big(A_t(B), (\omega_1^A)_B, (\omega_2^A)_B\big)$$

is an invertible map, and ℓ is as in (2.1.7). The equivalence class is denoted $[A_t, \alpha, \beta, \Psi]$. Composition in $\mathcal{SAB}_n^1(B)$ is defined as follows:

$$
\begin{array}{lcl}
(\tau_2, c_2) \circ (\tau_1, c_1) & = & (\tau_2\tau_1, \ell(\tau_1)^* c_2 \cdot c_1) \\
(\tau, c) \circ [A_t, \alpha, \beta, \Psi] & = & [A_t, \alpha, \beta\tau^{-1}, \ell(\tau^{-1})^* c \cdot \Psi] \\
[A_t, \alpha, \beta, \Psi] \circ (\tau, c) & = & [A_t, \alpha\tau, \beta, \Psi c] \\
[A_{t_2}, \alpha_2, \beta_2, \Psi_2] \circ [A_{t_1}, \alpha_1, \beta_1, \Psi_1] & = & [A_{t_1+t_2}, \iota_1\alpha_1, \iota_2\beta_2, \Psi_3].
\end{array}
$$

In e.g. the second case, the $C(S)^{\otimes -n}$-$C(S)^{\otimes -n}$-action on $F(A_t)^{\otimes -n}$ is induced from the bordism (not B-bordism!) given by $\ell(\alpha)$ and $\ell(\beta\tau^{-1})$. For the last case, ι_1, ι_2 are maps such that the square in the following diagram (1-forms suppressed),

$$
\begin{tikzcd}
& & A_{t_1+t_2}(B) & & \\
& A_{t_1}(B) \arrow[ur, "\iota_1"] & & A_{t_2}(B) \arrow[ul, "\iota_2"'] & \\
S(B) \arrow[ur, "\alpha_1"] & & S(B) \arrow[ul, "\beta_1"'] \arrow[ur, "\alpha_2"] & & S(B), \arrow[ul, "\beta_2"']
\end{tikzcd}
$$

is a pushout; Ψ_3 is the image of $\Psi_2 \otimes \Psi_1$ under the isomorphism (2.1.8), which is induced by applying ℓ to the above diagram.

The decorated B-isomorphisms in $\mathcal{SAB}_n^1(B)$ are classified by (2.1.4) and the fact that $C(S)^{\otimes -n} \cong Cl_{-n}$. The relations among decorated B-isomorphisms are given by

$$(1, c) \circ (\epsilon, 1) = (\epsilon, 1) \circ (1, \epsilon^* c), \tag{2.1.9}$$

$$(1, c) \circ (\tau_x, 1) = (\tau_x, 1) \circ (1, c). \tag{2.1.10}$$

Given a decorated B-bordism $(A_t, \alpha, \beta, \Psi)$, α and β are elements of (2.1.5), and there exists a unique element κ of (2.1.6) such that $\kappa\alpha = \nu_{0,0,0}$, while $\kappa\beta$ can be any element of (2.1.5) with $y^0 > 0$. Therefore, all the equivalence classes of decorated B-bordisms are precisely as follows:

$$A_{x,y,\theta,\Psi} = [A_{y^0}, \nu_{0,0,0}, \nu_{x,y,\theta}, \Psi], \qquad (\epsilon, 1) \circ A_{x,y,\theta,\Psi} = [A_{y^0}, \nu_{0,0,0}, \nu_{x,y,\theta}\epsilon, \Psi],$$

where $(x, y, \theta) \in \Lambda^{\text{even}}/\mathbb{Z} \times \Lambda^{\text{even}}_{(0,\infty)} \times \Lambda^{\text{odd}}$ and $\Psi \in F(A_{y^0})$. Let us define $A_{x,y,\theta} := A_{x,y,\theta,\Omega^{\otimes -n}}$. Notice that any $A_{x,y,\theta,\Psi}$ can be rewritten as $(1, c_1) \circ A_{x,y,\theta} \circ (1, c_2)$, for some $c_1, c_2 \in C(S)^{\otimes -n}$. The relations between decorated B-isomorphisms and decorated B-bordisms are given by

$$
\begin{aligned}
(\tau_{x_1}, 1) \circ A_{x_2,y,\theta} &= A_{x_2,y,\theta} \circ (\tau_{x_1}, 1) = A_{-x_1+x_2,\, y,\, \theta}, && (2.1.11)\\
(\epsilon, 1) \circ A_{x,y,\theta} &= A_{x,y,-\theta} \circ (\epsilon, 1), && (2.1.12)\\
(1, c) \circ A_{x,y,\theta} &= A_{x,y,\theta} \circ (1, c). && (2.1.13)
\end{aligned}
$$

The first two follow from simple calculations and the last one from the fact that $\lambda\Omega = \Omega\lambda$. Finally, a diagram similar to the one that proves (1.1.20) shows

$$A_{x_2,y_2,\theta_2} A_{x_1,y_1,\theta_1} = A_{x_1+x_2-\frac{1}{2}\theta_1\theta_2,\, y_1+y_2-\frac{i}{2}\theta_1\theta_2,\, \theta_1+\theta_2}. \tag{2.1.14}$$

Now, given a map of super manifolds $f : B' \to B$, the functor $\mathcal{SAB}^1_n(f) : \mathcal{SAB}^1_n(B) \to \mathcal{SAB}^1_n(B')$ is defined by

$$S \mapsto S, \quad (\epsilon, 1) \mapsto (\epsilon, 1), \quad (\tau_x, c) \mapsto (\tau_{f^*x}, c), \quad A_{x,y,\theta} \mapsto A_{f^*x, f^*y, f^*\theta},$$

where $f^* : \Lambda \to \Lambda' = \mathcal{O}(B')$. We indeed have a functor because the morphisms of $\mathcal{SAB}^1_n(B)$ are generated by $(\epsilon, 1)$ and those of the forms (τ_x, c) and $A_{x,y,\theta}$. This finishes the definition of the super category $\mathcal{SAB}^1_n$.

Definition 2.1.15. An *annular field theory, or AFT, of degree n* is a functor $E : \mathcal{SAB}^1_n \to \mathcal{S}\mathrm{Hilb}^{\mathbb{C}}$ of super categories. The super category $\mathcal{S}\mathrm{Hilb}^{\mathbb{C}}$ is defined similarly as $\mathcal{S}\mathrm{Hilb}$, but with complex Hilbert spaces and complex-linear operators. Let $E_B : \mathcal{SAB}^1_n(B) \to \mathcal{S}\mathrm{Hilb}(B)$ be the (ordinary) functors comprising E, where B is any object of $\mathcal{S}$. Since the values of E_B on S, $(\epsilon, 1)$ and $(1, c)$ are independent of B, [23] we simply write $\mathcal{H} := E(S)$, $E(\epsilon, 1)$ and $E(1, c)$. Fix B and let $\Lambda = \mathcal{O}(B)$. E_B is required to satisfy the following assumptions:

(i) $E_B(A_{x,y,\theta}) \in \Lambda \otimes B(\mathcal{H})$ are Hilbert-Schmidt operators.

(ii) $E_B(\tau_x, 1)^\dagger = E_B(\tau_{-x}, 1)$ and $E_B(A_{x,y,\theta})^\dagger = E_B(A_{-x,y,-i\theta})$. [24]

(iii) $\epsilon = E(\epsilon, 1)$ defines the $\mathbb{Z}/2$-grading in $\mathcal{H}$, i.e. $\epsilon|_{\mathcal{H}^{\text{even}}} = 1$, $\epsilon|_{\mathcal{H}^{\text{odd}}} = -1$.

(iv) $E_B(\tau_x, c)$ is analytic in x and linear in c. $E_B(A_{x,y,\theta,\Psi})$ is analytic in (x, y, θ) and linear in Ψ.

Let us analyze the consequences of the definition of an AFT E:

- By (iii) and (2.1.9), the operators $E(1, c)$, for $c \in C(S)^{\otimes -n} \cong Cl_{-n}$, define a $\mathbb{Z}/2$-graded Cl_{-n}-action on $\mathcal{H}$.
- By (ii) and $\tau_x\tau_{-x} = 1$, $x \in \mathbb{R}/\mathbb{Z} \mapsto E_B(\tau_x, 1)$ defines a unitary S^1-action on $\mathcal{H}$, so that $E_B(\tau_x, 1) = e^{2\pi i x P}$, where P is even, self-adjoint, Cl_{-n}-linear, because of (2.1.10). This extends to all $x \in \Lambda^{\text{even}}/\mathbb{Z}$ because of (iv).

[23] See definition 1.1.22.

[24] We make this assumption on $E_B(A_{x,y,\theta})^\dagger$ so that $E_B(A_{x,y,0})^\dagger = E_B(A_{-x,y,0})$, as expected according to the usual axioms of field theories, and taking adjoints respects (2.1.14). To check the latter, one should beware that $[(\theta T)(\theta' T')]^\dagger = -(\theta' T')^\dagger(\theta T)^\dagger$, for $T, T' \in B(\mathcal{H})$.

- Since (i), (ii) and (2.1.14) together imply $y \mapsto E_B(A_{0,y,0})$ is a commutative semigroup of self-adjoint, Hilbert-Schmidt operators, we have $E_B(A_{0,y,0})|_{\mathcal{H}'} = e^{-2\pi y H}$ and $E_B(A_{0,y,0})|_{\mathcal{H}'^{\perp}} = 0$, where $\mathcal{H}' \subset \mathcal{H}$ is a subspace and H is a self-adjoint operator on $\mathcal{H}'$ with compact resolvent. Again, this holds for all $y \in \Lambda^{\mathrm{even}}_{(0,\infty)}$. Furthermore, due to (2.1.12) and (2.1.13), $\mathcal{H}'$ is in fact a $\mathbb{Z}/2$-graded Cl_{-n}-submodule, and H is even and Cl_{-n}-linear. For simplicity, we just write $E_B(A_{0,y,0}) = e^{-2\pi y H}$, thinking that '$H = +\infty$' on $\mathcal{H}'^{\perp}$.

- The relation in (2.1.11) implies that P and H commute.

- Let $E_B(A_{0,y,\theta}) = e^{-2\pi y H} + \sqrt{2\pi i}\theta K(y)$. It follows from (2.1.12), (ii) and (2.1.13) that $K(y)$ is odd, self-adjoint and Cl_{-n}-linear. Applying E_B to (2.1.14) with $x_1 = x_2 = 0$ and using (2.1.11), we obtain

$$\begin{aligned} &\sqrt{2\pi i}\left[\theta_1 e^{-2\pi y_2 H} K(y_1) + \theta_2 K(y_2) e^{-2\pi y_1 H}\right] + 2\pi i \theta_1\theta_2 K(y_2)K(y_1) \\ = \;& \sqrt{2\pi i}(\theta_1 + \theta_2) K(y_1 + y_2 - \tfrac{i}{2}\theta_1\theta_2) + \pi i \theta_1\theta_2 (H + P). \end{aligned} \tag{2.1.16}$$

  Putting $\theta_1 = 0$ and $\theta_2 = 0$ into (2.1.16) separately shows

$$K(y_2)e^{-2\pi y_1 H} = K(y_1 + y_2) = e^{-2\pi y_2 H} K(y_1).$$

  In particular, all $K(y)$ commute with H. Let $\tilde{G} = K(1)e^{2\pi H}$. For $y > 1$, we have $K(y) = K(1)e^{-2\pi(y-1)H} = \tilde{G}e^{-2\pi y H}$; this in fact extends to all $y \in \Lambda^{\mathrm{even}}_{(0,\infty)}$ by (iv). Finally, putting this back into (2.1.16) yields $2\tilde{G}^2 = H + P$.

To summarize, let $q = e^{2\pi i(x+iy)}$ and $\bar{q} = e^{-2\pi i(x-iy)}$; using (2.1.11), we have

$$E_B(A_{x,y,\theta}) = q^L \bar{q}^{\tilde{G}^2}(1 + \sqrt{2\pi i}\theta\tilde{G}), \tag{2.1.17}$$

[25] $\forall (x,y,\theta) \in \Lambda^{\mathrm{even}}/\mathbb{Z} \times \Lambda^{\mathrm{even}}_{(0,\infty)} \times \Lambda^{\mathrm{odd}}$, where $L = -P + \tilde{G}^2$ is even, $\tilde{G}$ is odd, and both are self-adjoint, Cl_{-n}-linear and have compact resolvent. Furthermore, $L - \tilde{G}^2 = -P$ has integer eigenvalues, and the eigenvalues of $L + \tilde{G}^2 = H$ are bounded below. The pair $(L, \tilde{G})$ are called the *generators* of E.

A few words about AFTs. Our definitions of $\mathcal{SAB}_n$, and hence of AFTs, are unsatisfactory for a number of reasons. Obviously, we would like to include bordisms given by more general surfaces. Furthermore, these surfaces should be equipped with an appropriate notion of 'super conformal structure.' The author has not been able to identify such a structure. The geometric structures attached to the super manifolds S and A_t in the definition of $\mathcal{SAB}_n$ should be particular representatives of certain 'super conformal classes.' Also, our choices of the Clifford algebra $C(S)$ and the Fock spaces $F(A_t)$, though will give us the desired result below, are rather ad hoc.

2.2. Categories $\mathcal{AFT}_n$

For any AFT E, we denote its generators by $(L_E, \tilde{G}_E)$.

Definition 2.2.1. Let $n \in \mathbb{Z}$, and

$$\mathcal{H} = \bigoplus_{k\in\mathbb{Z}} \mathcal{H}_k = \bigoplus_{k\in\mathbb{Z}} (\mathcal{H}_k^{\mathrm{even}} \oplus \mathcal{H}_k^{\mathrm{odd}})$$

[25] More conceptually, an AFT is determined by two operators because the B-bordisms, as B varies, define a super semigroup by (2.1.14), and its super Lie algebra is generated by two elements.

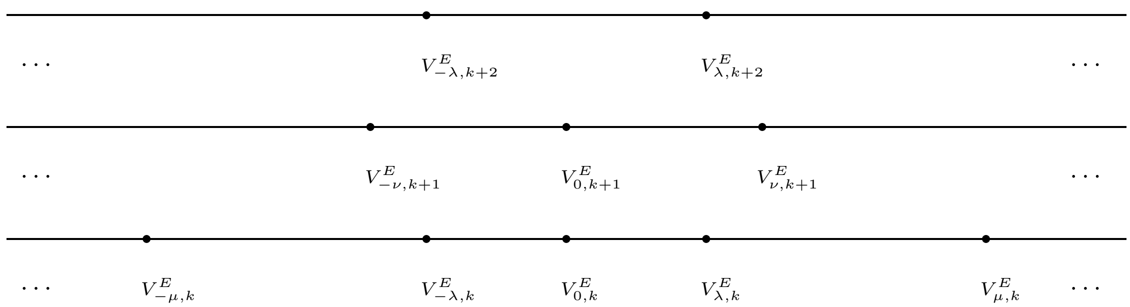

Figure 4: The spectral decomposition of $(\tilde{G}_E, L_E - \tilde{G}_E^2)$ for an AFT E.

be a $\mathbb{Z}\times\mathbb{Z}/2$-graded Hilbert space, such that each $\mathcal{H}_k$ admits a stable (see footnote 14) $\mathbb{Z}/2$-graded Cl_{-n}-action. We define a topological category $\mathcal{AFT}_n(\mathcal{H})$ as follows. An object of $\mathcal{AFT}_n(\mathcal{H})$ is an AFT E of degree n such that $E(S) = \mathcal{H}$, $\mathcal{H}_k = \{L_E - \tilde{G}_E^2 = k\}$ and the given graded Cl_{-n}-action on $\mathcal{H}$ coincides with the one coming from E. In oder to define the morphisms in $\mathcal{AFT}_n(\mathcal{H})$, we need to first discuss the spectral decompositions of the generators of an AFT of degree n.

Recall from (2.1.17) that any AFT E is determined by its generators $(L_E, \tilde{G}_E)$, which have the properties that (i) L_E is even, $\tilde{G}_E$ is odd, (ii) L_E and $\tilde{G}_E$ commute and are self-adjoint, (iii) they are Cl_{-n}-linear, (iv) they have compact resolvent, (v) $L_E - \tilde{G}_E^2$ has integer eigenvalues, and (vi) L_E is bounded below on $\ker \tilde{G}_E$. By (ii), the operators L_E and $\tilde{G}_E$ have real spectra and a simultaneous spectral decomposition. Furthermore, (iv) implies their spectra are discrete and each eigenspace is finite dimensional. Let $V^E_{\lambda,k}$ be the common eigenspace of $\tilde{G}_E$, with eigenvalue λ, and $L_E - \tilde{G}_E^2$, with eigenvalue $k \in \mathbb{Z}$. By (iii), $V^E_{\lambda,k}$ are Cl_{-n}-submodules of $\mathcal{H}_k$, and (vi) says $V^E_{0,k} = 0$ for $k << 0$. Finally, (i) implies $\epsilon V^E_{\lambda,k} = V^E_{-\lambda,k}$, where $\epsilon|_{\mathcal{H}^{\text{even}}} = 1$ and $\epsilon|_{\mathcal{H}^{\text{odd}}} = -1$. Figure 4 shows a picture of these spectral data. In the picture, the lines represent the subset $\mathbb{R}\times\mathbb{Z}$ of $\mathbb{R}^2$. We regard $\mathrm{sp}(\tilde{G}_E, L_E - \tilde{G}_E^2)$, the simultaneous spectrum of $\tilde{G}_E$ and $L_E - \tilde{G}_E^2$, as a subset of $\mathbb{R}\times\mathbb{Z}\subset\mathbb{R}^2$, and label each of its points (λ, k) with the corresponding eigenspace $V^E_{\lambda,k}$. As in the case of 1-SEFTs, there is a symmetry about the y-axis in figure 4 due to the isomorphisms between $V^E_{\lambda,k}$ and $V^E_{-\lambda,k}$ defined by ϵ.

Now, we can define the morphisms in $\mathcal{AFT}_n(\mathcal{H})$. A morphism from an object E to another object E' is a 'deformation' (α, f, A) from the spectral decomposition of $(\tilde{G}_E, L_E - \tilde{G}_E^2)$ to that of $(\tilde{G}_{E'}, L_{E'} - \tilde{G}_{E'}^2)$, where

- $\alpha : \mathrm{sp}(\tilde{G}_E, L_E - \tilde{G}_E^2) \to \mathrm{sp}(\tilde{G}_{E'}, L_{E'} - \tilde{G}_{E'}^2)$ is a proper map that is constant on the second coordinate, odd and order preserving in the first coordinate (regarding sp $(\tilde{G}_E, L_E - \tilde{G}_E^2)$, etc., as subsets of $\mathbb{R}^2$),
- $f : \bigoplus_{(\lambda,k)} V^E_{\lambda,k} \to \bigoplus_{(\lambda',k')} V^{E'}_{\lambda',k'}$ is an even map that embeds $V^E_{\lambda,k}$ isometrically into $V^{E'}_{\alpha(\lambda,k)}$, and
- $A \subset \mathcal{H}$ is a Cl_{-n}-submodule, such that $\tilde{G}_{E'} \geq 0$ on A, $A \perp \epsilon A$ and $\bigoplus_{\lambda'} V^{E'}_{\lambda'} = f(\bigoplus_\lambda V^E_\lambda) \oplus A \oplus \epsilon A$.

Fixing α and $f \in \prod_{(\lambda,k)} \mathrm{Hom}(V^E_{\lambda,k}, V^{E'}_{\alpha(\lambda,k)})$, A is determined by $A \cap V^{E'}_{0,k} \in \mathrm{Gr}(V^{E'}_{0,k})$, $k \in \mathbb{Z}$. Indeed, $A \oplus \epsilon A$ is the orthogonal complement of the image of f, and the assumption that $\tilde{G}_{E'} \geq 0$ determines A up to the

subspace where $\tilde{G}_{E'} = 0$. The set of morphisms $\{(\alpha, f, A)\}$ from E to E' is topologized as a subspace of

$$\coprod_{\alpha} \left(\prod_{(\lambda,k)\in \mathrm{sp}(\tilde{G}_E, L_E - \tilde{G}_E^2)} \mathrm{Hom}(V^E_{\lambda,k}, V^{E'}_{\alpha(\lambda,k)}) \times \prod_{k\in\mathbb{Z}} \mathrm{Gr}(V^{E'}_{0,k}) \right).$$

Composition of morphisms is defined by

$$(\alpha', f', A') \circ (\alpha, f, A) = (\alpha'\alpha, f'f, f'(A) \oplus A').$$

This finishes the definition of $\mathcal{AFT}_n(\mathcal{H})$.

Theorem 2.2.2. *For each $n \in \mathbb{Z}$, we have*

$$|\mathcal{AFT}_n(\mathcal{H})| \simeq (K_{Tate})_n.$$

The right hand side is the n-th space of the elliptic spectrum K_{Tate} associated with the Tate curve.

We briefly recall the Tate curve and K_{Tate}. For each $q \in \mathbb{C}$, $|q| < 1$, there is an elliptic curve over $\mathbb{C}$ defined by a Weierstrass equation

$$y^2 + xy = x^3 + a_4(q)x + a_6(q).$$

If $0 < |q| < 1$, the elliptic curve is isomorphic to $\mathbb{C}^\times/q^{\mathbb{Z}}$ as a Riemann surface, with the group structure induced from multiplication in $\mathbb{C}^\times$. As q varies, a_4 and a_6 define analytic functions over the open disk $\{\,|q| < 1\}$. Their Taylor series at $q = 0$ have integer coefficients, hence the above equation also defines an elliptic curve C_{Tate} over $\mathbb{Z}[[q]]$, the ring of integral formal power series. [Sil94] However, we will instead regard C_{Tate} as an elliptic curve over $\mathbb{Z}[[q]][q^{-1}]$, the ring of integral formal Laurent series.

An elliptic spectrum (E, C, t) consists of an even periodic ring spectrum E, an elliptic curve C over $\pi_0 E$, and an isomorphism $t : \mathrm{Spf}\, E^0(\mathbb{C}P^\infty) \xrightarrow{\sim} \widehat{C}$ of formal groups over $\pi_0 E$. [AHS01] On one hand, as a result of even periodicity, $E^0(\mathbb{C}P^\infty)$ is abstractly the ring of formal power series over $\pi_0 E$, and the formal scheme $\mathrm{Spf}\, E^0(\mathbb{C}P^\infty)$ admits a formal group structure. On the other hand, the group structure on C induces a formal group structure on $\widehat{C}$, the formal neighborhood of C at the identity. As an example, the elliptic spectrum associated with the Tate curve is

$$K_{\mathrm{Tate}} = (K[[q]][q^{-1}], C_{\mathrm{Tate}}, t_{\mathrm{Tate}}),$$

where the cohomology theory associated with $K[[q]][q^{-1}]$ is periodic K-theory tensored with $\mathbb{Z}[[q]][q^{-1}]$. The formal groups arising from $K[[q]][q^{-1}]$ and C_{Tate} are both the multiplicative formal group.

Proof of theorem 2.2.2. Recall the categories $\mathcal{V}_n$ from section 1.3. For each $k \in \mathbb{Z}$, define a functor $\mathrm{ind}_{n,k} : \mathcal{AFT}_n(\mathcal{H}) \to \mathcal{V}_n(\mathcal{H}_k)$ by $E \mapsto V^E_{0,k}$ and

$$(\alpha, f, A) \mapsto \Big(f|_{V^E_{0,k}}, f\big(\bigoplus_{\substack{\lambda>0 \\ \alpha(\lambda,k)=(0,k)}} V^E_{\lambda,k}\big) \oplus (A \cap V^{E'}_{0,k}) \Big).$$

Collecting $\mathrm{ind}_{n,k}$ for all $k \in \mathbb{Z}$ yields another functor,

$$\mathrm{ind}_n^{S^1} : \mathcal{AFT}_n(\mathcal{H}) \to {\prod_{k\in\mathbb{Z}}}' \mathcal{V}_n(\mathcal{H}_k).$$

The symbol $\prod'$ denotes the full subcategory of the product category consisting of objects of the form $(V_k)_{k\in\mathbb{Z}}$, where $V_k \in N_0\mathcal{V}_n(\mathcal{H}_k)$ and $V_k = 0$ for all $k << 0$. (Recall: $V^E_{0,k} = 0$ for $k << 0$ for any AFT E.)

On the other hand, there is a functor in the opposite direction

$$\mathcal{E}_n : \prod_{k\in\mathbb{Z}}{}' \mathcal{V}_n(\mathcal{H}_k) \to \mathcal{AFT}_n(\mathcal{H}).$$

An object of $\prod'_k \mathcal{V}_n(\mathcal{H}_k)$ is a sequence $\mathbf{V} = (V_k)_{k\in\mathbb{Z}}$ of $\mathbb{Z}/2$-graded Cl_{-n}-modules. The degree n AFT $\mathcal{E}_n(\mathbf{V})$ is defined by $\mathcal{E}_n(\mathbf{V})(S) = \mathcal{H}$ and

$$\mathcal{E}_n(\mathbf{V})_B(A_{x,y,\theta}) = \begin{cases} q^k & \text{on } V_k \\ 0 & \text{on } (\bigoplus_k V_k)^\perp \end{cases},$$

where B is any object of $\mathcal{S}$ and $q = e^{2\pi i(x+iy)}$. For a morphism $(\mathbf{f}, \mathbf{A}) = (f_k, A_k)_{k\in\mathbb{Z}}$ in $\prod'_k \mathcal{V}_n(\mathcal{H}_k)$, say from $\mathbf{V} = (V_k)$ to $\mathbf{V}' = (V'_k)$, we have

$$\mathcal{E}_n((\mathbf{f}, \mathbf{A})) = \left(\alpha_{\mathbf{V},\mathbf{V}'}, \bigoplus_{k\in\mathbb{Z}} f_k, \bigoplus_{k\in\mathbb{Z}} A_k\right),$$

where $\alpha_{\mathbf{V},\mathbf{V}'}$ is the inclusion of $\{(0,k)|V_k \neq 0\}$ into $\{(0,k)|V'_k \neq 0\}$. Notice that (i) by definition, $f_k : V_k \to V'_k$ are embeddings, so $V'_k \neq 0$ if $V_k \neq 0$, and (ii) we have $\mathrm{sp}(\tilde{G}_{\mathcal{E}_n(\mathbf{V})}, L_{\mathcal{E}_n(\mathbf{V})} - \tilde{G}^2_{\mathcal{E}_n(\mathbf{V})}) = \{(0,k)|V_k \neq 0\}$, etc.

The composition $\mathrm{ind}^{S^1}_n \circ \mathcal{E}_n$ is the identity on $\prod'_k \mathcal{V}_n(\mathcal{H}_k)$. Also, for each object E of $\mathcal{AFT}_n(\mathcal{H})$, there is a morphism

$$(\alpha_E, i_E, A_E) : \mathcal{E}_n(\mathrm{ind}^{S^1}_n(E)) \to E,$$

where α_E is the inclusion $\{(0,k) : V_k \neq 0\} \subset \mathrm{sp}(\tilde{G}_E, L_E - \tilde{G}^2_E)$, i_E is the inclusion $\bigoplus_k V^E_{0,k} \subset \bigoplus_{(\lambda,k)} V^E_{\lambda,k}$, and $A_E = \bigoplus_{\lambda>0} V^E_{\lambda,k}$. One easily checks that every diagram

$$\begin{tikzcd}
\mathcal{E}_n(\mathrm{ind}^{S^1}_n(E)) \arrow[r, "(\alpha_E, i_E, A_E)"] \arrow[d, "\mathcal{E}_n(\mathrm{ind}^{S^1}_n(\alpha,f,A))"'] & E \arrow[d, "(\alpha,f,A)"] \\
\mathcal{E}_n(\mathrm{ind}^{S^1}_n(E')) \arrow[r, "(\alpha_{E'}, i_{E'}, A_{E'})"] & E'
\end{tikzcd}$$

commutes. Therefore, $E \mapsto (\alpha_E, i_E, A_E)$ is a natural transformation from $\mathcal{E}_n \circ \mathrm{ind}^{S^1}_n$. This proves the functors $\mathrm{ind}^{S^1}_n$ and $\mathcal{E}_n$ induce a homotopy equivalence

$$|\mathcal{AFT}_n(\mathcal{H})| \simeq \left|\prod_k{}' \mathcal{V}_n(\mathcal{H}_k)\right| = \prod_k{}' |\mathcal{V}_n(\mathcal{H}_k)|,$$

where the last term is the space consisting of points $(x_k)_{k\in\mathbb{Z}}$ with $x_k \in |\mathcal{V}_n(\mathcal{H}_k)|$ and $x_k = 0$ (the zero submodule) for $k << 0$. By proposition 1.4.2, we have

$$\left[X, |\mathcal{AFT}_n(\mathcal{H})|\right] = \prod_k{}' K^n(X) = K^n(X)[[q]][q^{-1}],$$

where the second term consists of sequences $(a_k)_{k\in\mathbb{Z}}$ with $a_k \in K^n(X)$ and $a_k = 0$ for $k << 0$, and the second equality identifies such a sequence with the formal Laurent series $\sum_k a_k q^k$. $\square$

Appendix: Proofs of Two Lemmas

In this appendix, we give the proofs of lemmas 1.1.15 and 1.1.27. For any map $f : M \to N$ of super manifolds, we will denote by $f_B : M(B) \to N(B)$ the natural transformation $g \mapsto f \circ g$, where B is any super manifold.

Lemma 1.1.15. *Suppose (M_i, χ_i), $i = 1, 2$, are Euclidean $(0|1)$- or $(1|1)$-manifolds, and B is a super manifold with $B_{red} = \{b\}$. Given a map*

$$\phi : \big(M_1(B), (\chi_1)_B\big) \to \big(M_2(B), (\chi_2)_B\big),$$

there exists a unique map $\phi_{red} : (M_1)_{red} \to (M_2)_{red}$ of ordinary manifolds so that the diagram

$$\begin{tikzcd} \big(M_1(B), (\chi_1)_B\big) \arrow[r, "\phi"] \arrow[d] & \big(M_2(B), (\chi_2)_B\big) \arrow[d] \\ (M_1)_{red} \arrow[r, "\phi_{red}"] & (M_2)_{red} \end{tikzcd},$$

with the vertical arrows given by $f \mapsto f_{red}(b)$, commutes. In other words, $\phi(f)_{red} = \phi_{red} \circ f_{red}$, $\forall f \in M_1(B)$. Furthermore, $(\phi'\phi)_{red} = \phi'_{red} \circ \phi_{red}$ whenever ϕ and ϕ' are composable.

Proof. Assume that $(M_i)_{\mathrm{red}}$, $i = 1, 2$, are connected. This is sufficient because, for any super manifold N, the map $f \mapsto f_{\mathrm{red}}(b)$ sends different components of $N(B)$ to different components of N_{red}.

The lemma is clear for $\dim M_1 = \dim M_2 = (0|1)$.

Suppose $\dim M_1 = (0|1)$ and $\dim M_2 = (1|1)$. Since $(M_1)_{\mathrm{red}}$ is a point, the existence of ϕ_{red} is equivalent to the map $f \mapsto \phi(f)_{\mathrm{red}}(b)$, $f \in M_1(B)$, being constant, or just locally constant, as $M_1(B)$ is connected. Fix $f_0 \in M_1(B)$ and let $y = \phi(f_0)_{\mathrm{red}}(b)$. If $\lambda_1 \in \mathcal{O}(M_1)^{\mathrm{odd}}$ is an element such that $\chi_1 = \lambda_1 d\lambda_1$, sending λ to λ_1 defines an invertible map $\mu : (M_1, \chi_1) \xrightarrow{\sim} (\mathbb{R}^{0|1}, \lambda d\lambda)$. On the other hand, there exists a neighborhood U of $y \in (M_2)_{\mathrm{red}}$, together with an invertible map $\tau : ((M_2)_U, \chi_2|_{(M_2)_U}) \xrightarrow{\sim} (\mathbb{R}^{1|1}_{(a_1,a_2)}, ds + \lambda d\lambda)$, for some $a_1 < a_2$. (See the note below (1.1.5).) Since $\phi(f_0) \in (M_2)_U(B) \subset M_2(B)$, let $V \subset M_1(B)$ be a neighborhood of f_0 such that $\phi(V) \subset (M_2)_U(B)$. Also, let $V' = \mu_B(V) \subset \mathbb{R}^{0|1}(B)$. We have the following commutative diagram:

$$\begin{tikzcd} \big(V, (\chi_1)_B\big) \arrow[r, "\phi|_V"] \arrow[d, "\mu_B"', "\cong"] & \big((M_2)_U(B), (\chi_2|_{(M_2)_U})_B\big) \arrow[d, "\tau_B"', "\cong"] \\ \big(V', (\lambda d\lambda)_B\big) \arrow[r, "\tilde{\phi}"] & \big(\mathbb{R}^{1|1}_{(a,a')}(B), (ds + \lambda d\lambda)_B\big). \end{tikzcd}$$

By (1.1.12), $\tilde{\phi} = \gamma_{z,\theta}$ or $\gamma_{z,\theta}\epsilon$, for some $(z, \theta) \in \mathcal{O}(B)^{\mathrm{even}} \times \mathcal{O}(B)^{\mathrm{odd}}$. For any $\eta \in V'$,

$$\tilde{\phi}(\eta)_{\mathrm{red}}(b) = \gamma_{z,\theta}(\pm\eta)_{\mathrm{red}}(b) = z^0 \in (a_1, a_2),$$

which is independent of η. (Recall that z^0 denotes the unique real number such that $z - z^0$ is nilpotent.) This means $\phi(f)_{\mathrm{red}}(b) = y$ for any $f \in V$, as desired.

Now, suppose $\dim M_1 = \dim M_2 = (1|1)$. Fix $x \in (M_1)_{\text{red}}$. Let

$$M_1(B)_x = \{f \in M_1(B) | f_{\text{red}}(b) = x\}.$$

We need to show that the map $f \mapsto \phi(f)_{\text{red}}(b)$ is constant on $M_1(B)_x$. Let $U \subset (M_1)_{\text{red}}$ now be a neighborhood of x and $s_U \in \mathcal{O}((M_1)_U)^{\text{even}}$, $\lambda_U \in \mathcal{O}((M_1)_U)^{\text{odd}}$ be local functions such that $\chi_1|_{(M_1)_U} = ds_U + \lambda_U d\lambda_U$. Define a map $\gamma : (\mathbb{R}^{0|1}, \lambda d\lambda) \to (M_1, \chi_1)$ by $\gamma_{\text{red}}(\mathbb{R}^0) = x$, $\gamma^*(\lambda_U) = \lambda$. Notice that any $f \in M_1(B)_x$ is in the image of $\gamma_B : \mathbb{R}^{0|1}(B) \to M_1(B)$; indeed, viewing $f^*(\lambda_U)$ as an element of $\mathbb{R}^{0|1}(B)$ using (1.1.10), we have $\gamma_B(f^*(\lambda_U)) = f$. As proved above, the composition

$$\big(\mathbb{R}^{0|1}(B), (\lambda d\lambda)_B\big) \xrightarrow{\gamma_B} \big(M_1(B), (\chi_1)_B\big) \xrightarrow{\phi} \big(M_2(B), (\chi_2)_B\big) \to (M_2)_{\text{red}} ,$$

where the last map is $g \mapsto g_{\text{red}}(b)$, is constant. This proves ϕ is constant on $M_1(B)_x$.

Given

$$\begin{aligned} \phi &: \big(M_1(B), (\chi_1)_B\big) \to \big(M_2(B), (\chi_2)_B\big), \\ \phi' &: \big(M_2(B), (\chi_1)_B\big) \to \big(M_3(B), (\chi_3)_B\big), \end{aligned}$$

where (M_i, χ_i), $i = 1, 2, 3$, are Euclidean $(0|1)$- or $(1|1)$-manifolds, we have

$$(\phi'\phi)_{\text{red}} f_{\text{red}} = (\phi'\phi)(f)_{\text{red}} = \phi'(\phi(f))_{\text{red}} = \phi'_{\text{red}}\phi(f)_{\text{red}} = \phi'_{\text{red}}\phi_{\text{red}} f_{\text{red}},$$

for any $f \in M_1(B)$. Any $x \in (M_1)_{\text{red}}$ is the image of f_{red} for some $f \in M_1(B)$, hence we have $(\phi'\phi)_{\text{red}}(x) = \phi'_{\text{red}}\phi_{\text{red}}(x)$, $\forall x \in (M_1)_{\text{red}}$. □

Note. We need not consider $(\dim M_1, \dim M_2) = ((1|1), (0|1))$, because in this case,

$$\operatorname{Hom}\big(M_1(B), (\chi_1)_B; M_2(B), (\chi_2)_B\big) = \emptyset.$$

It suffices and is easy to check this for $(M_1, \chi_1) = (\mathbb{R}^{1|1}_{(a_1,a_2)}, ds + \lambda d\lambda)$, where $a_1 < a_2$, and $(M_2, \chi_2) = (\mathbb{R}^{0|1}, \lambda d\lambda)$.

Lemma 1.1.27. *Suppose (M_i, χ_i), $i = 1, 2$, are Euclidean $(0|1)$- or $(1|1)$-manifolds and B is a super manifold with B_{red} = a point. There is a map*

$$\ell : \mathit{Hom}\big(M_1(B), (\chi_1)_B; M_2(B), (\chi_2)_B\big) \to \mathit{Hom}(M_1, \chi_1; M_2, \chi_2),$$

with the following properties:

(i) $\ell(\phi)_{red} = \phi_{red}$ *for any* $\phi \in \mathit{Hom}\big(M_1(B), (\chi_1)_B; M_2(B), (\chi_2)_B\big)$,

(ii) $\ell(\varphi_B) = \varphi$ *for any* $\varphi \in \mathit{Hom}(M_1, \chi_1; M_2, \chi_2)$, *and*

(iii) $\ell(\phi'\phi) = \ell(\phi')\ell(\phi)$ *whenever* ϕ *and* ϕ' *are composable.*

Proof. First, consider the case in which each (M_i, χ_i), $i = 1, 2$, is either $(\mathbb{R}^{0|1}, \lambda d\lambda)$ or $(\mathbb{R}^{1|1}_{(a_1,a_2)}, ds + \lambda d\lambda)$, where $-\infty \le a_1 < a_2 \le +\infty$. Recall (1.1.3)-(1.1.5) and (1.1.11)-(1.1.13). We define

$$\begin{aligned} \ell(1) = 1, &\quad \ell(\epsilon) = \epsilon, \\ \ell(\gamma_{z,\theta}) = \gamma_{z^0}, &\quad \ell(\gamma_{z,\theta}\epsilon) = \gamma_{z^0}\epsilon, \\ \ell(\tau_{z,\theta}) = \tau_{z^0}, &\quad \ell(\tau_{z,\theta}\epsilon) = \tau_{z^0}\epsilon, \end{aligned} \tag{A.3}$$

for any $(z,\theta) \in \mathcal{O}(B)^{\text{even}} \times \in \mathcal{O}(B)^{\text{odd}}$. (Again, z^0 is the unique real number such that $z - z^0$ is nilpotent.) One easily verifies (A.3) satisfies (i)-(iii).

In general, consider a map $\phi : \big(M_1(B), (\chi_1)_B\big) \to \big(M_2(B), (\chi_2)_B\big)$. Let $U_i \subset (M_i)_{\text{red}}$, $i = 1, 2$, be open sets, such that $\phi_{\text{red}}(U_1) \subset U_2$ and there are invertible maps φ_i from $\big((M_i)_{U_i}, \chi_i|_{(M_i)_{U_i}}\big)$ to either $(\mathbb{R}^{0|1}, \lambda d\lambda)$ or $(\mathbb{R}^{1|1}_{(a_1,a_2)}, ds + \lambda d\lambda)$ for some $a_1 < a_2$. According to the definition of ϕ_{red} in lemma 1.1.15, ϕ maps $(M_1)_{U_1}(B)$ into $(M_2)_{U_2}(B)$. Using (A.3), we define

$$\ell(\phi|_{(M_1)_{U_1}(B)}) = \varphi_2^{-1} \circ \ell\big((\varphi_2)_B \circ \phi|_{(M_1)_{U_1}(B)} \circ (\varphi_1)_B^{-1}\big) \circ \varphi_1. \tag{A.4}$$

It follows from properties (ii), (iii) for (A.3) that the right hand side is independent of the choices of φ_i. This implies the maps $\ell(\phi|_{(M_1)_{U_1}(B)})$ for various U_1 agree on overlaps, and thus glue into a single map $\ell(\phi) : (M_1, \chi_1) \to (M_2, \chi_2)$.

It suffices to verify (i)-(iii) locally, i.e. for (A.4). For example, $\ell(\phi|_{(M_1)_{U_1}(B)})_{\text{red}} = (\phi|_{(M_1)_{U_1}(B)})_{\text{red}}$ follows from lemma 1.1.15, the fact that (A.3) satifies (i), and $((\varphi_i)_B)_{\text{red}} = (\varphi_i)_{\text{red}}$, $i = 1, 2$. The proofs of (ii) and (iii) are similar. $\square$

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